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(Mathematical Sciences)

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A. RAMANATHAN

(1946–1993)

We regret to inform our readers that Professor Annamalai Ramanathan, who was a co-editor of this journal during the last year and currently a member of its Editorial Board, passed away on 12 March 1993, at the young age of 46, following a heart attack.

Professor Ramanathan was an accomplished mathematician, a Fellow of the Academy and a recipient of the Shanti Swarup Bhatnagar Prize. He was a fine person, very helpful to colleagues and students, and it was a pleasure to exchange ideas with him. His death is a major loss to this journal.

—Editor

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Transformation formula for exponential sums involving Fourier coefficients of modular forms

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Abstract. In 1984 Jutila [5] obtained a transformation formula for certain exponential sums involving the Fourier coefficients of a holomorphic cusp form for the full modular group $SL(2, \mathbb{Z})$. With the help of the transformation formula he obtained good estimates for the distance between consecutive zeros on the critical line of the Dirichlet series associated with the cusp form and for the order of the Dirichlet series on the critical line, [7]. In this paper we follow Jutila to obtain a transformation formula for exponential sums involving the Fourier coefficients of either holomorphic cusp forms or certain Maass forms for congruence subgroups of $SL(2, \mathbb{Z})$ and prove similar estimates for the corresponding Dirichlet series.

Keywords. Exponential sums; summation formula; cusp forms and Maass forms; transformation formula.

1. Introduction

By an exponential sum we mean a sum of the form

$$\sum_{M \leq n \leq M'} a(n)e(f(n)),$$

where $a(n)$ is an arithmetical function and f is a real valued function on $[M, M']$. Many a problem in number theory reduces in the ultimate analysis to estimating an exponential sum. The Waring's problem, the Goldbach's conjecture, the Dirichlet divisor problem and the order of the Dirichlet series in the critical strip are very good examples of this phenomenon.

The most commonly employed method to estimate such sums is due to Van der Corput and Vornoi. The basic idea here is to transform an exponential sum into a new shape by first converting the sum into an integral—Van der Corput's lemma and Vornoi summation formula—and then evaluating the integral by the 'Saddle-point method'. Since then various summation formulae of the Voronoi type have been found; a very good survey is to be found in [2]. In 1984, Jutila [5] discovered that by replacing $f(n)$ by $f(n) + rn$ where r is an integer (which does not affect the sum) before transforming the sum one was led to much better transformed sums. Another important observation made by Jutila was the flexibility of this method which, he showed, works in the case when $a(n)$'s are Fourier coefficients of a cusp form of weight k for the full modular group, $SL(2, \mathbb{Z})$. With the help of this transformation formula he was able to obtain for the Dirichlet series associated to cusp forms for $SL(2,\mathbb{Z})$ analogues of many results known in the case of the Riemann zeta function, $\zeta(s)$, like the distance between consecutive zeros on the critical line, the order on the critical line, mean square estimates and higher power moments [5, 6, 7]. In this paper we show that the transformation formula and the above mentioned applications carry over to the case when $a(n)$'s are Fourier coefficients of either holomorphic cusp forms or certain Maass forms (Maass forms f with $\Delta f = 1/4f$) of higher levels. While some of the above mentioned applications were already known in the case of cusp forms for $SL(2,\mathbb{Z})$ due to Good [4] they seem to be new in the case of cusp forms of higher levels.

Mention must also be made of the work of Meurman [8, 9] who has extended some of the results of Jutila to the case of L-functions associated to Maass wave forms for $SL(2,\mathbb{Z})$. Presumably his work also extends to higher level Maass forms. The class of Maass forms we consider in this thesis does not occur at level one.

2. Functional equations and summation formulae

In this section we are concerned with functional equations for Dirichlet series (and their twists by additive characters) associated to certain modular forms for congruence subgroups of the full modular group $SL(2,\mathbb{Z})$. The class of modular forms we consider consists of all holomorphic forms and the subclass of Maass forms with 1\4 as the eigenvalue of the Laplacian on the upper half plane. We consider the holomorphic case (Theorem 2.1) and the case of Maass forms (Theorem 2.2) separately. The summation formulae these functional equations lead to are written down at the end of the section.

If f is a function on the upper half-plane $\mathbb{H}$, k an integer and A is in $GL^+(2,\mathbb{R})$ then $f|^k_{[A]}(\tau)$ will denote the following function

$$(\det A)^{k/2}(c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) \quad \text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Holomorphic case

For k, $N \geqslant 1$ integers and ε a character $\operatorname{mod} N$ let $M(N,k,\varepsilon)$ denote the space of modular forms of level N, weight k and character ε. Thus if $f \in M(N,k,\varepsilon)$ and $A \in \Gamma_0(N)$ we have

$$f|^k_{[A]}(\tau) = \varepsilon(d) f(\tau), \quad \text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note that $M(N,k,\varepsilon) = \{0\}$ unless $\varepsilon(-1) = (-1)^k$, and that $M(N,k,1)$ is $M(N,k)$, the space of modular forms of weight k for $\Gamma_0(N)$. If $H(N)$ denotes the matrix $\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ then $f \to f|^k_{[H(N)]}$ defines an isomorphism from $M(N,k,\varepsilon)$ onto $M(N,k,\bar{\varepsilon})$ where $\bar{\varepsilon}$ is the (complex) conjugate character.

Let $f \in M(N,k,\varepsilon)$ and $f(\tau) = \sum_{n=0}^{\infty} a(n) \exp(2\pi i n\tau)$ be its Fourier expansion at the cusp $i\infty$. We are interested in functional equations for the Dirichlet series

$$\phi_f(s, h/m) = \sum_{n=1}^{\infty} \frac{a(n)\exp(2\pi i nh/m)}{n^s}.$$

Theorem 2.1. (a) *Case when* $(m,N)=N$. *Let*

$$\Phi(s,h/m)=(m/2\pi)^s\Gamma(s)\phi_f(s,h/m).$$

Then

$$\Phi(s,h/m)+\frac{a(0)}{s}+\frac{i^k\varepsilon(\bar h)a(0)}{k-s},$$

is EBV (entire and bounded in every vertical strip) and we have the functional equation:

$\Phi(s,h/m)=i^k\varepsilon(\bar h)\Phi(k-s,-\bar h/m)$, *where* $\bar h$ *is defined by* $h\bar h\equiv 1 \pmod m$.

(b) *Case when* $(m,N)=1$. *Let* $f|^k_{[H(N)]}(\tau)=g(\tau)\in M(N,k,\bar\varepsilon)$ *and* $g(\tau)=\sum_{n=0}^{\infty} b(n)\exp(2\pi in\tau)$ *be it's Fourier expansion. Further let*

$\Phi(s,h/m)=(m\sqrt{N}/2\pi)^s\Gamma(s)\phi_f(s,h/m)$ *and*

$\Psi(s,h/m)=(m\sqrt{N}/2\pi)^s\Gamma(s)\phi_g(s,h/m)$.

Then

$$\Phi(s,h/m)+(m^2N)^{-s/2}\left(\frac{a(0)}{s}+\frac{\varepsilon(m)b(0)}{k-s}\right)$$

is EBV and we have the functional equation:

$$\Phi(s,h/m)=\varepsilon(m)\Psi(k-s,-\overline{Nh}/m). \qquad \square$$

Proof. (a) Let $t>0$ be a real number and put $\tau=h/m+i/mt$ and $\tau'=-\bar h/m+it/m$. Then τ and τ' are in the upper half-plane and are equivalent under $\Gamma_0(N)$ by the matrix (remember $m\equiv 0 \pmod N$)

$$A=\begin{bmatrix} h & (\bar h h-1)/m \\ m & \bar h \end{bmatrix}$$

i.e. $A(\tau')=\tau$. Therefore we have $f(\tau)=\varepsilon(\bar h)(it)^k f(\tau')$. If $\mathrm{Re}(s)$ is sufficiently large we have

$$\begin{aligned}\Phi(s,h/m)&=\sum_{n=1}^{\infty} a(n)\exp(2\pi inh/m)\int_0^\infty t^{s-1}\exp(-2\pi nt/m)\,\mathrm{d}t.\\ &=\int_0^\infty t^{s-1}\{f(h/m+it/m)-a(0)\}\,\mathrm{d}t.\\ &=\int_1^\infty t^{s-1}\{f(h/m+it/m)-a(0)\}\,\mathrm{d}t-\int_0^1 a(0)t^{s-1}\\ &\quad+\int_0^1 t^{s-1}f(h/m+it/m)\,\mathrm{d}t.\end{aligned}$$

Now consider

$$\begin{aligned}\int_0^1 t^{s-1}f(h/m+it/m)\,\mathrm{d}t&=\int_1^\infty t^{-s-1}f(h/m+i/mt)\,\mathrm{d}t.\\ &=\int_1^\infty t^{-s-1}f(\tau)\,\mathrm{d}t.\\ &=\varepsilon(\bar h)i^k\int_1^\infty t^{k-s-1}f(\tau')\,\mathrm{d}t.\end{aligned}$$

Thus we have

$$\Phi(s,h/m)=\int\{[f(h/m+it/m)-a(0)]t^{s-1}$$
$$+\varepsilon(\bar{h})i^k[f(-\bar{h}/m+it/m)-a(0)]t^{k-s-1}\}\mathrm{d}t$$
$$-\frac{a(0)}{s}-\frac{\varepsilon(\bar{h})i^k a(0)}{k-s}.$$

This integral representation proves the claims made in (a).
(b). For $x\in\mathbb{R}$ let $\alpha(x)$ denote the matrix $\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)$. Let $t>0$ be a real number and set $\tau=h/m+i/m^2Nt$ and $\tau'=-\bar{h}/m+it$. We need to know $f(\tau)$ in terms of $g(\tau')$. For that first observe that

$$\alpha(h/m)H(m^2N)=H(N)\begin{pmatrix} m & -b\\ -Nh & n\end{pmatrix}\alpha(b/m)\begin{pmatrix} m & 0\\ 0 & m\end{pmatrix}$$

where b is defined via $Nhb\equiv -1 \pmod{m}$ (which is possible because $(m,N)=(m,h)=1$.) and n is chosen such that $\begin{pmatrix} m & -b\\ -Nh & n\end{pmatrix}$ is in $\Gamma_0(N)$. Therefore we have

$$f(\tau)=(m^2N)^{k/2}(it)^k f|_{[\alpha(h/m)][H(Nm^2)]}(\mathrm{it}).$$
$$=(m^2N)^{k/2}(it)^k f|_{[H(N)][(\begin{smallmatrix} m & -b\\ -Nh & n\end{smallmatrix})][\alpha(b/m)]}(it).$$
$$=(m^2N)^{k/2}(it)^k\bar{\varepsilon}(n)g|_{[\alpha(b/m)]}(it).$$
$$=(m^2N)^{k/2}(it)^k\varepsilon(m)g(-\bar{N}\bar{h}/m+it).$$

Note that we have made use of

(i) $\bar{\varepsilon}(n)=\varepsilon(m)$ as $mn\equiv 1 \pmod{N}$,

and

(ii) $\exp(2\pi ib/m)=\exp(-2\pi i\bar{N}\bar{h}/m)$

since

$Nhb\equiv -1 \pmod{m}$.

Consider now the following integral representation:

$$\Phi(s,h/m)=(m^2N)^{s/2}\int_0^\infty [f(h/m+it)-a(0)]t^{s-1}\mathrm{d}t.$$
$$=(m^2N)^{s/2}\left[\int_{1/m\sqrt{N}}^\infty [f(h/m+it)-a(0)]t^{s-1}\mathrm{d}t-\int_0^{1/m\sqrt{N}} a(0)t^{s-1}\mathrm{d}t\right.$$
$$\left.+\int_0^{1/m\sqrt{N}} f(h/m+it)t^{s-1}\mathrm{d}t\right].$$

Now

$$\int_0^{1/m\sqrt{N}} f(h/m+it)t^{s-1}\mathrm{d}t=(m^2N)^{-s}\int_{1/m\sqrt{N}}^\infty f(h/m+i/m^2Nt)t^{-s-1}\mathrm{d}t,$$
$$=(m^2N)^{(k/2)-s}\varepsilon(m)\int_{1/m\sqrt{N}}^\infty g(-\bar{N}\bar{h}/m+it)t^{k-s-1}\mathrm{d}t.$$

Thus we get

$$\begin{aligned}\Phi(s,h/m)=(m^2N)^{s/2}\int_{1/m\sqrt{N}}^{\infty}&\{[f(h/m+it)-a(0)]t^{s-1}\\&+\varepsilon(m)(m^2N)^{(k/2)-s}[g(-N\bar{h}/m+it)-b(0)]t^{k-s-1}\}\,dt\\&-(m^2N)^{-s/2}\left(\frac{a(0)}{s}+\frac{\varepsilon(m)b(0)}{k-s}\right).\end{aligned}$$

This integral representation verifies the claims made in (b). Thus the proof of the Theorem is complete.

Remark. As these functional equations characterise modular forms of level N (see [12] and [13]) we cannot in general hope to get similar functional equations when $1<(m,N)<N$. For instance if f is a new form of level N then existence of functional equations for $1<(m,N)<N$ would mean that f is a form of lower level which it is not.

Non-holomorphic case (see [11])

Let $\Delta=-y^2(\partial^2/\partial x^2+\partial^2/\partial y^2)$ denote the Laplacian on the upper half-plane $\mathbb{H}$ associated with the hyperbolic metric. Let $N\geqslant 1$ be an integer, ε a character mod N and λ a complex number. Let f be an even Maass form of level N, character ε and eigenvalue (for Δ) λ. This means:

(i) $f\in L^2(\Gamma_1(N)\backslash\mathbb{H})$;
(ii) $f(\gamma\tau)=\varepsilon(d)f(\tau)$ for $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(N)$;
(iii) $\Delta f=\lambda f$, $\lambda=1/4+r^2$;
(iv) f is a simultaneous eigenfunction of the Hecke operators T_n,

$$(n,N)=1 \text{ and } T_{-1}f(\tau)=f(-\bar{\tau})=f(\tau).$$

Such an f has a Fourier expansion of the following type:

$$f(\tau)=\sum a(n)y^{1/2}K_{ir}(2\pi ny)\cos 2\pi nx,\quad \tau=x+iy.$$

Let f be an even Maass form with $\lambda=1/4$ i.e. $r=0$ with Fourier coefficients $a(n)$. Here again we are interested in functional equations for the following Dirichlet series associated with f:

$$\phi_f(s,h/m)=\sum_{n=1}^{\infty}\frac{a(n)\cos 2\pi nh/m}{n^s}$$

$$\phi'_f(s,h/m)=\sum_{n=1}^{\infty}\frac{a(n)\sin 2\pi nh/m}{n^s}$$

and in this direction we have:

Theorem 2.2. (a) $(m,N)=N$. *The functional equations are*

$$\Phi_f(s,h/m)=\varepsilon(h)\Phi_f(1-s,-\bar{h}/m)$$

$$\Phi'_f(s,h/m)=-\varepsilon(h)\Phi'_f(1-s,-\bar{h}/m),$$

where $h\bar{h} \equiv 1 \pmod m$

$$\Phi_f(s, h/m) = (m/\pi)^s \Gamma^2(s/2) \phi_f(s, h/m),$$

and

$$\Phi'_f(s, h/m) = (m/\pi)^s \Gamma^2\left(\frac{s+1}{2}\right) \phi'_f(s, h/m)$$

(b) $(m, N) = 1$. *Let* $f|^0_{[H(N)]}(\tau) = g(\tau)$; *then* g *is an even Maass form of level* N *and character* $\bar{\varepsilon}$. *Let* $b(n)$ *be the Fourier coefficients of* g. *Then the functional equations are:*

$$\Phi_f(s, h/m) = \varepsilon(m)\Phi_g(1-s, -\bar{N}\bar{h}/m),$$

$$\Phi'_f(s, h/m) = \varepsilon(m)\Phi'_g(1-s, -\bar{N}\bar{h}/m)$$

where

$$\Phi_f(s, h/m) = (\pi/m\sqrt{N})^{-s} \Gamma^2(s/2) \phi_f(s, h/m)$$

and

$$\Phi'_f(s, h/m) = (\pi/m\sqrt{N})^{-s} \Gamma^2\left(\frac{s+1}{2}\right) \phi'_f(s, h/m).$$ □

Proof. Let $f_x(\tau) = 1/2\pi i\, \partial/\partial x\, f(\tau)$; $(\tau = x + iy)$. Then we have

$$f_x(\tau) = i \sum a(n) n \sqrt{y} K_0(2\pi n y) \sin(2\pi n x).$$

We need to know how $f_x(\tau)$ transforms under the transformations A and $H(N)$. For this let $f_y = 1/2\pi i\, \partial/\partial y\, f(\tau)$. If $U \in \Gamma_0(N)$ then $f(U\tau) = \varepsilon(d) f(\tau)$, $U = \begin{pmatrix} a\, b \\ d\, c \end{pmatrix}$. So we get

$$\begin{aligned} f_x(\tau) &= 1/2\pi i\, \partial/\partial x\, f(\tau) = \varepsilon(d)/2\pi i\, \partial/\partial x\, f(Ut) \\ &= \varepsilon(d)[f_x(U\tau)\mathrm{Re}(\partial U\tau/\partial x) + f_y(U\tau)\mathrm{Im}(\partial U\tau/\partial x)] \\ &= \varepsilon(d)[f_x(U\tau)\mathrm{Re}((c\tau + d)^{-2}) + f_y(U\tau)\mathrm{Im}((c\tau + d)^{-2}] \end{aligned}$$

Now taking $\tau = h/m + i/mt$ and A as in the above proof we see that $f_x(h/m + i/mt) = \varepsilon(\bar{h})(-t)^{-2} f_x(-\bar{h}/m + it/m)$. We get a similar transformation formula under H(N).

The rest of the proof is along the same lines as that of theorem 2.1 (and so we will not reproduce it here) but will use the following formula to get an integral representation for $\Phi(s, h/m)$:

$$1/4(m/\pi n)^s \Gamma^2(s/2) = \int_0^\infty K_0(2\pi n y/m) y^{s-1} \, dy.$$

Summation formulae

We begin by stating a theorem of Berndt [3]. Let $\{\lambda_n\}$ and $\{\mu_n\}$-be two sequences of positive numbers strictly increasing to infinity. Let $\{a(n)\}$ and $\{b(n)\}$ be two sequences of complex numbers, not identically zero such that the Dirichlet series:

$$\phi(s) = \sum a(n) \lambda_n^{-s} \quad \text{and} \quad \psi(s) = \sum b(n) \mu_n^{-s}$$

converge in some half-plane. Suppose further that they satisfy the functional equation:

$\chi(s)\,\phi(s) = \chi(r-s)\psi(r-s)$ where $\chi(s)$ is one of the following three gamma factors:

(i) $\Gamma(s)$ and r arbitrary real;
(ii) $\Gamma(s/2)\Gamma(s-p/2)$ where p is an integer and $r = p+1$;
(iii) $\Gamma^2(s+1/2)$ and $r=1$.

Also further, suppose that the poles of $\chi(s)\phi(s)$ are confined to some compact set. Define for $x>0$, $Q_q(x)$ and $I_q(x)$ as follows:

$$Q_q(x) = \frac{1}{2\pi i}\int_{C_q} \frac{\Gamma(s)\phi(s)}{\Gamma(s+q+1)} x^{s+q} ds$$

where C_q is a cycle enclosing all of the integrand's poles; and respectively as $\chi(s)$ is as in (i), (ii) and (iii):

$$\begin{aligned} I_q(x) &= x^{(r+q)/2} J_{r+q}(2\sqrt{x}) \\ &= x^{(p+q+1)/2}\Big\{\cos(\pi(p+1)/2) J_{p+q+1}(4\sqrt{x}) \\ &\qquad - \sin(\pi(p+1)/2)\Big[Y_{p+q+1}(4\sqrt{x}) + \left(\frac{2(-1)^{p+q}}{\pi}\right)K_{p+q+1}(4\sqrt{x})\Big\} \\ &= x^{(q+1)/2}\Big\{Y_{q+1}(4\sqrt{x}) + \left(\frac{2(-1)^{q+1}}{\pi}\right)K_{q+1}(4\sqrt{x})\Big\}. \end{aligned}$$

Theorem 2.3. (Berndt [3]): *Let $f \in C^{(1)}(0,\infty)$. Then*

$$\sum_{a<\lambda_n<b}{}' a(n) f(\lambda_n) = \int_a^b Q_0'(t) f(t)\mathrm{d}t + \sum \frac{b(n)}{\mu_n^{r-1}} \int_a^b I_{-1}(\mu_n t) f(t)\mathrm{d}t \qquad \square$$

Summation formulae for the holomorphic case

For the sake of simplicity we write down the summation formulae only when f is a cusp form. In this case only the second term on the right hand side of the general summation formula in theorem 2.3 will survive for then $\phi(s)$ is entire. Accordingly let f be a cusp form of level N and character ε. The functional equations of theorem 2.1 give rise to the following summation formulae:

(a). *Case when $(m,N) = N$.*

$$\sum_{a\leqslant n\leqslant b} a(n)\exp(2\pi inh/m) f(n) = i^k \varepsilon(\bar{h})\frac{2\pi}{m}\sum a(n)\exp(-2\pi in\bar{h}/m) n^{-(k-1)/2} \times \int_a^b x^{(k-1)/2} J_{k-1}\left(\frac{4\pi(nx)^{1/2}}{m}\right) f(x)\mathrm{d}x$$

(b). *Case when $(m,N) = 1$.*

$$\sum_{a\leqslant n\leqslant b} a(n)\exp(2\pi inh/m) f(n) = \bar{\varepsilon}(m)\left(\frac{2\pi}{m\sqrt{N}}\right)\sum b(n)\exp(-2\pi i\bar{N}\bar{h}/m) n^{-(k-1)/2} \int_a^b x^{(k-1)/2} J_{k-1}\left(\frac{4\pi(nx)^{1/2}}{m\sqrt{N}}\right) f(x)\mathrm{d}x.$$

Summation formulae for the non-holomorphic case

Let f be an even Maass form with $\lambda = 1/4$ with Fourier coefficients $a(n)$. Then the functional equations of theorem 2.2 imply the following summation formulae (note that $p = 0$):

(a). $(m, N) = N$.

$$\sum_{a\leqslant n\leqslant b} a(n)\cos(2\pi nh/m)f(n) = \frac{\varepsilon(\bar{h})\pi}{2m}\sum a(n)\cos(-2\pi n\bar{h}/m)\times$$
$$\int_a^b\left[2/\pi K_0\left(\frac{4\pi\sqrt{(nx)}}{m}\right) - Y_0\left(\frac{4\pi\sqrt{(nx)}}{m}\right)\right]f(x)\mathrm{d}x,$$

$$\sum_{a\leqslant n\leqslant b} a(n)\sin(2\pi nh/m)f(n) = \frac{\varepsilon(\bar{h})\pi}{2m}\sum a(n)\sin(-2\pi n\bar{h}/m)\times$$
$$\int_a^b\left[Y_0\left(\frac{4\pi\sqrt{(nx)}}{m}\right) + 2/\pi K_0\left(\frac{4\pi\sqrt{(nx)}}{m}\right)\right]f(x)\mathrm{d}x,$$

(b). $(m, N) = 1$.

$$\sum_{a\leqslant n\leqslant b} a(n)\cos(2\pi nh/m)f(n) = \frac{\varepsilon(m)\pi}{2m\sqrt{N}}\sum b(n)\cos(-2\pi n\bar{N}\bar{h}/m)\times$$
$$\int_a^b\left[2/\pi K_0\left(\frac{4\pi(nx)^{1/2}}{m\sqrt{N}}\right) - Y_0\left(\frac{4\pi(nx)^{1/2}}{m\sqrt{N}}\right)\right]f(x)\mathrm{d}x,$$

$$\sum_{a\leqslant n\leqslant b} a(n)\sin(2\pi nh/m)f(n) = \frac{\varepsilon(m)\pi}{2m\sqrt{N}}\sum b(n)\sin(-2\pi\bar{N}\bar{h}/m)\times$$
$$\int_a^b\left[Y_0\left(\frac{4\pi(nx)^{1/2}}{m\sqrt{N}}\right) + 2/\pi K_0\left(\frac{4\pi(nx)^{1/2}}{m\sqrt{N}}\right)\right]f(x)\mathrm{d}x.$$

3. Transformation formulae

In this section we obtain, following Jutila [6], a transformation formula for exponential sums of the type

$$\sum_{a\leqslant n\leqslant b} a(n)g(n)\exp(2\pi i f(n))$$

where $a(n)$'s are Fourier coefficients of either a holomorphic cusp form or a Maass form with $\lambda = 1/4$ for a congruence subgroup of $SL(2, \mathbb{Z})$ and f and g are functions on $[a, b]$. To begin with we recall results on exponential integrals due to Atkinson and Jutila. This is essentially chapter 2 of [6] without proofs. Then we obtain transformation formulae for exponential sums involving Fourier coefficients of cusp forms considered in §1. We conclude this section with some special cases of the transformation formula for Dirichlet polynomials associated with these cusp forms.

Exponential integrals

Let

$$I=\int_a^b g(x)\exp(2\pi i(f(x)+\alpha x))\,\mathrm{d}x=\int_a^b h(x)\,\mathrm{d}x.$$

For a positive integer J and a positive real number U define a smoothed version I_J of I by:

$$U^{-1}\int_0^U \mathrm{d}u_1\int_0^U \mathrm{d}u_J\int_{a+u}^{b-u} h(x)\,\mathrm{d}x=\int_a^b \eta_J(x)h(x)\,\mathrm{d}x,\quad u=u_1+\cdots+u_J.$$

Also let $I_0=I$. Note that $0<\eta(x)\leqslant 1$ for $x\in(a,b)$ and $\eta(x)=1$, for $a+JU\leqslant x\leqslant b-JU$.

We quote three theorems below first of which gives an approximate value of the integral I in terms of saddle points (Atkinson), the second theorem its generalization to I_J due to Jutila and the third gives an estimate of I_J when f has no saddle points in (a,b). For proofs of these theorems see [6].

Let f and g be functions on $[a,b]$ satisfying the following conditions:

(i) f is real for $a\leqslant x\leqslant b$;
(ii) f and g are holomorphic in the domain
$D=\{z\,|\,|z-x|<\mu$ for some $x\in(a,b)\}$ where μ is a positive real number;
(iii) there are positive numbers F and G such that:
$|g(z)|\ll G$ and $|f'(z)|\ll F\mu^{-1}$ for $z\in D$;
(iv) $f''(x)>0$ and $f''(x)\gg F\mu^{-2}$.

Since $f''(x)>0$, $f'(x)+\alpha$ is monotonically increasing and hence has at most one zero in (a,b), say x_0. Further let

$$E_J(x)=G(|f'(x)+\alpha|+f''(x)^{1/2})^{-J-1}$$

Theorem 3.1. *Let f and g be as above. Then*

$$\begin{aligned}I&=g(x_0)f''(x_0)^{-1/2}\exp[(2\pi i(f(x_0)+\alpha x_0+1/8)]\\&\quad+O(G\exp[-A|\alpha|\mu-AF](b-a))+O(G\mu F^{-3/2})+O(E_0(a))+O(E_0(b)).\end{aligned}$$

□

Theorem 3.2. *Let $U>0, J\geqslant 0$ be a fixed integer, $JU<(b-a)/2$ and f and g be as above with the additional condition that $F\gg 1$. Suppose, also that $U\gg \mu F^{-1/2}$. Then with I_J as above we have:*

$$\begin{aligned}I_J&=\xi(x_0)g(x_0)f''(x_0)^{-1/2}\exp(f(x_0)+\alpha x_0+1/8)\\&\quad+O((1+(\mu/U)^J)G\exp(-A|\alpha|\mu-AF)(b-a))\\&\quad+O((1+F^{1/2})G\mu F^{-3/2})\\&\quad+O\left(U^{-J}\sum_{j=0}^{J}(E_J(a+jU)+E_J(b-jU)\right),\end{aligned}$$

where $\xi(x_0) = 1$ *for* $a + JU < x_0 < b - JU$,

$$\xi(x_0) = (J!U^J)^{-1} \sum_{j=0}^{J1} \binom{J}{j} (-1)^j \sum_{0 \leqslant \nu \leqslant J/2} c_\nu f''(x_0)^{-\nu} (x_0 - a - jU)^{J-2\nu}$$

for $a < x_0 \leqslant a + JU$ *with* j_1 *the largest integer such that* $a + j_1 U < x_0$

$$\xi(x_0) = (J!U^J)^{-1} \sum_{j=0}^{J2} \binom{J}{j} (-1)^j \sum_{0 \leqslant \nu \leqslant J/2} c_\nu f''(x_0)^{-\nu} (b - x_0 - jU)^{J-2\nu}$$

for $b - JU \leqslant x_0 < b$ *with* j_2 *the largest integer such that* $b - j_2 U > x_0$. *The* c_ν *are numerical constants.* □

Theorem 3.3. *Suppose* f *and* g *are functions satisfying* (*i*) *and* (*ii*) *above. Assume further that* $|g(z)| \ll G$, $|f'(x)|M$, *and* $|f'(z)| \ll M$ *for* $z \in D$ *and* $x \in (a,b)$. *Let* I_J *denote the smoothed version of* I *with* $\alpha = 0$ *and* $0 < JU < (b-a)/2$. *Then*

$$I_J \ll U^{-1} G M^{-J-1} + (\mu^J U^{1-J} + (b-a)) G \exp(-AM\mu).$$ □

Transformation formulae

Before we proceed to the theorem we quote a Lemma (without proof) which summarizes the properties of Hankel functions we need to use.

Lemma 3.4 *Let* $\delta_1 < \pi$ *and* δ_2 *be fixed positive numbers. Then in the sector* $|\arg z| \leqslant \pi - \delta_1$, $|z| \geqslant \delta_2$ *we have*

$$H_n^{(j)}(z) = (2/\pi z)^{1/2} \exp((-1)^{j-1} i(z - n\pi/2 - \pi/4))(1 + g_j(z)),$$

where the functions $g_j(z)$ *are holomorphic in the slit complex plane* $z \neq 0$, $|\arg z| < \pi$, *and satisfy* $|g_j(z)| \ll |z|^{-1}$ *in the above sector. Further we have*

$$J_n(z) = 1/2(H_n^{(1)}(z) + H_n^{(2)}(z))$$

and

$$Y_n(z) = 1/2i(H_n^{(1)}(z) - H_n^{(2)}(z)).$$

We also have

$$K_n(x) = (\pi/2x)^{1/2} \exp(-x)(1 + O(x^{-1})).$$ □

In what follows, δ denotes an arbitrary small positive constant not necessarily the same in each occurrence. Put $L = \log M$.

We have the following theorem which gives a transformation formula in the case of *holomorphic cusp forms*. Accordingly let

$$f(\tau) = \sum_{n=1}^{\infty} a(n) \exp(2\pi i n \tau)$$

be a cusp form of level N, weight k and character ε. Further let $f|^k_{[H(N)]}(\tau) = g(\tau) = \Sigma b(n) \exp(2\pi i n \tau)$.

Theorem 3.5. *Let* $2 \leqslant M_1 < M_2 \leqslant 2M_1$ *and let* f *and* g *be holomorphic functions in the domain*

$$D = \{z \mid |z - x| < cM_1 \text{ for some } x \in [M_1, M_2]\},$$

where c *is a positive constant. Suppose that* $f(x)$ *is real for* x *in* $[M_1, M_2]$. *Suppose also that, for some positive numbers* F *and* G

$$|g(z)| \ll G,$$

$$|f'(z)| \ll FM^{-1}, \text{ for } z \in D, \text{ and that } (0<) f''(x) \gg FM_1^{-2} \text{ for } x \in [M_1, M_2].$$

Let $r = h/m$ *be a rational number such that*

$$1 \leqslant m \ll M_1^{1/2-\delta},$$

$$|r| \asymp FM_1^{-1},$$

and

$$f'(M(r)) = r$$

for a certain number $M(r)$ *in* $[M_1, M_2]$. *Write* $M_j = M(r) + (-1)^j m_j$, $j = 1, 2$.

Suppose that $m_1 \asymp m_2$, *and that*

$$M_1^{\delta} \max(M_1 F^{-1/2}, |hm|) \ll m_1 \ll M_1^{1-\delta}.$$

Define for $j = 1, 2$

$$p_{j,n}(x) = \begin{array}{ll} f(x) - rx + (-1)^{j-1}\left(\dfrac{2\sqrt{(nx)}}{m} - \dfrac{(k-1)}{4} - \dfrac{1}{8}\right), & \text{if } (m, N) = N; \text{ and} \\ f(x) - \mathrm{r}x + (-1)^{j-1}\left(\dfrac{2\sqrt{(nx)}}{m\sqrt{N}} - \dfrac{(k-1)}{4} - \dfrac{1}{8}\right), & \text{if } (m, N) = 1. \end{array}$$

and

$$n_j = \begin{cases} (r - f'(M_j))^2 m^2 M_j, & \text{if } (m, N) = N; \\ (r - f'(M_j))^2 (m\sqrt{N})^2 M_j & \text{if } (m, N) = 1. \end{cases}$$

and for $n < n$ *let* $x_{j,n}$ *be the (unique) zero of* $p'_{j,n}(x)$ *in the interval* (M_1, M_2). *Set*

$$A = \begin{cases} i^k \varepsilon(\bar{h})(2m)^{-1/2}, & \text{if } (m, N) = N \text{ and} \\ \varepsilon(m)(2m\sqrt{N})^{-1/2}, & \text{if } (m, N) = 1. \end{cases}$$

Then we have

$$\sum_{M_1 \leqslant n \leqslant M_2} a(n) g(n) e(f(n)) =$$

$$A \sum_{j=1}^{2} (-1)^{j-1} \sum_{n < n_j} a'(n) e(nh'/m) n^{-(k/2)+1/4} x_{j,n}^{(k/2)-(3/4)} g(x_{j,n}) \times$$

$$p''_{j,n}(x_{j,n})^{-1/2} e(p_{j,n}(x_{j,n}) + 1/8) + O(G(|h|m)^{1/2} M_1^{(k-1)/2} m_1^{1/2} L^2) +$$

$$O(F^{1/2} G |h|^{-3/4} m^{5/4} M_1^{(k-1)/2} m_1^{-1/4} L).$$

where $a'(n) = a(n)$, $h' = -\bar{h}$ *if* $(m, N) = N$ *and* $a(n) = b(n)$, $h' = -\bar{N}\bar{h}$ *if* $(m, N) = 1$. □

Proof. Without loss of generality suppose that $r(=h/m)$ is positive. Assume that $(m, N) = N$; the case $(m, N) = 1$ is entirely similar. The transformation formula should be understood as an asymptotic result wherein M_1 and M_2 are large. Before we start on the proof, we shall note various estimates that are needed; like, for instance, the order of n_j. First note that $f''(x) \asymp FM^{-2}$ (for, by assumption, $f''(x) \gg FM^{-2}$ and the reverse inequality follows from the estimate for f' and holomorphy of f), and $F \gg M_1^{1/2+\delta}$ (for $F \gg M_1 r \geqslant m^{-1} M_1 \gg M_1^{1/2+\delta}$). Thus we have

$$|r - f'(M_j)| \asymp m_j F M_1^{-2} \text{ (for } f'(M(r)) = r).$$

This gives us the estimate: $n \asymp F^2 m^2 M_1^{-3} m_j^2$.

The n_j's are determined by the condition $p'_{j,n}(M_j) = 0$. This implies that for $n < n_j, p'_{j,n}(x)$ has an unique zero in (M_1, M_2). For clearly $(-1)^j p'_{j,n}(M_j) > 0$ and $(-1)^j p'_{j,n}(M(r)) < 0$, if $n < n_j$. Note also that $x_{1,n} < x_{2,n}$ and that $p'_{j,n}(x)$ has no zero in (M_1, M_2) if $n > n_j$. Uniqueness of $x_{j,n}$ follows from that $p''_{j,n}(x)$ has the same order as $f''(x)$ if M_1 is sufficiently large and hence is positive for $f''(x)$ is positive.

Let

$$S = S(M_1, M_2) = \sum_{M_1 \leqslant n \leqslant M_2} a(n) g(n) e(f(n)).$$

We first replace S by its smoothed version S':

$$S' = U^{-1} \int_0^U S(u) \mathrm{d}u,$$

where

$$S(u) = \sum_{M_1 + u \leqslant n \leqslant M_2 - u} a(n) g(n) e(f(n))$$

and U is a parameter to be chosen later. For now we only assume

$$M_1^\delta \ll U \leqslant 1/2 \min(m_1, m_2).$$

The estimate $a(n) \ll n^{(k-1)/2+\varepsilon}$ implies that $S - S' \ll GUM_1^{(k-1)/2} L$. The choice of the parameter U later will show that this error has been accounted for in the statement of the transformation formula.

The idea is to apply the summation formula to $S(u)$ and evaluate S' by using saddle-point theorems. But instead of applying the summation formula to $S(u)$ as it stands it has been observed by Jutila that we get better results if we introduce an exponential factor without disturbing the sum. Accordingly before applying the summation formula we modify the sum $S(u)$ as:

$$S(u) = \sum_{a \leqslant n \leqslant b} a(n) e(nr) g(n) e(f(n) - nr), \quad a = M_1 + u, \quad b = M_2 - u.$$

Applying the summation formula of § 2 we get:

$$S(u) = i^k \varepsilon(\bar{h}) \frac{2\pi}{m} \sum a(n) e(-n\bar{h}/m) n^{-(k-1)/2}$$
$$\times \int_a^b x^{(k-1)/2} J_{k-1}\left(\frac{4\pi\sqrt{(n\pi)}}{m}\right) g(x) e(f(x) - rx) \mathrm{d}x.$$

Now write $J_{k-1}()$ in terms of the Hankel functions to get:

$$S(u) = i^k \varepsilon(\bar{h}) \sum a(n) e(-n\bar{h}/m) n^{-(k-1)/2} I_n,$$

where

$$I_n = \frac{\pi}{m} \int_a^b x^{(k-1)/2} \left[H^{(1)}_{k-1}\left(\frac{4\pi\sqrt{(nx)}}{m}\right) + H^{(2)}_{k-1}\left(\frac{4\pi\sqrt{(nx)}}{m}\right) \right] g(x) e(f(x) - rx)\,\mathrm{d}x$$

whence by lemma 3.4 we get $I_n = I^{(1)}_n + I^{(2)}_n$ with

$$I^{(j)}_n = (2m\sqrt{n})^{-1/2} \int_a^b x^{(k/2)-(3/4)} g(x) \left[1 + g_j\left(\frac{4\pi\sqrt{(nx)}}{m}\right) \right] e(p_{j,n}(x))\,\mathrm{d}x.$$

It can checked that the conditions of the theorems 3.1 and 3.3 are satisfied with $-r$ in place of r and $f(x)$ replaced by:

$$f(x) + (-1)^{j-1}(2\sqrt{(nx)}/m - (k-1)/4 - 1/8), \quad \text{and } \mu = M_1.$$

The number $x_{j,n}$ is by definition the saddle point for $I^{(j)}_n$ and it lies in the interval $[M_1, M_2]$ if and only if $n < n_j$. However, in $I^{(j)}_n$ the interval of integration is $[a, b] = [M_1 + u,\ M_2 - u]$, and $x_{j,n} \in [a, b]$ if and only if $n < n_j(u)$ where

$$n_j(u) = (r - f'(M_j + (-1)^{j-1}u)^2 m^2 (M_j + (-1)^{j-1}u).$$

But for simplicity we count the saddle point terms for all $n < n_j$ for this frees the saddle point terms from depending on u and thus we will have the same saddle point terms for all $S(u)$ and hence for S' as well. The number of extra terms counted will be

$$\ll 1 + n_j - n_j(u) \ll 1 + F^2 m^2 M_1^{-3} m_1 U.$$

The saddle point term for $I^{(j)}_n$, for $n < n_j$ is:

$$(2m)^{-1/2} n^{-1/4} x_{j,n}^{(k/2)-(3/4)} g(x_{j,n}) p''_{j,n}(x_{j,n})^{-1/2}$$
$$e(p_{j,n}(x_{j,n}) + 1/8)(1 + g_j(2(nx_{j,n})^{1/2}/m).$$

Thus up to $g_j()$ we have the explicit terms claimed in the theorem. The effect of the omission of $g_j()$ is:

$$\ll F^{-1/2} G m^{1/2} M_1^{(k/2)-(1/4)} \sum_{n<n_j} a(n) n^{-(k/2)-(1/4)}$$
$$\ll F^{-1/2} G m^{1/2} M_1^{(k/2)-(1/4)} n_j^{1/4} \ll G m_1^{1/2} M_1^{(k-1)/2} L.$$

This error can be absorbed into the first O-term in the formula given in the theorem.

The extra saddle-points counted while replacing $n_j(u)$ by n_j contribute

$$\ll (1 + F^2 m^2 M_1^{-3} m_1 U) F^{-1/2} G m^{-1/2} M_1^{(k/2)+(1/4)+\varepsilon} n_1^{-1/4}$$
$$\ll F^{1/2} G h^{-3/2} m^{1/2} m_1^{-1/2} M_1^{(k-1)/2+\varepsilon} + F^{-1/2} G h^{3/2} m^{-1/2} m_1^{1/2} M_1^{(k-1)/2+\varepsilon} U.$$

Here the first term is absorbed into the second O-term in the transformation formula and later U will be chosen so that the second term above also goes into the second O-term.

We shall now consider the error terms of theorem 3.1 which was applied to $I_n^{(j)}$ for $n < n_j$. The first error term is clearly negligible. The contribution of the second O-term is:

$$\ll F^{-3/2} G m^{-1/2} M_1^{(k/2)+(1/4)} \sum_{n \ll n_1} a(n) n^{-(k-1)/2} n^{-1/4}$$

$$\ll G m m_1^{3/2} M_1^{(k-1)/2-(3/2)} L \ll G m_1^{1/2} M_1^{(k-1)/2} L$$

which again goes into the first O-term of the theorem. The terms $O(E_0(a))$ and $O(E_0(b))$ are similar and so it is enough to consider one of them, say $O(E_0(a))$. This error term is

$$\ll G m^{-1/2} M_1^{(k/2)-(3/4)} n^{-1/4} (|p'_{j,n}(a)| + p''_{j,n}(a)^{1/2})^{-1}.$$

Consider the case $j = 1$; the case $j = 2$ is even simpler for $p'_{2,n}(b)$ cannot be very small. $p'_{1,n(u)}(a) = 0$ and $p''_{1,n}(a) = F^{-1} r^2$. Therefore we have

$$(|p'_{1,n}(a)| + p''_{1,n}(a)^{1/2})^{-1} \ll \begin{cases} F^{1/2} r^{-1} \text{ for } |n - n_1(u)| \ll F^{-1/2} h^2 m_1, \\ m M_1^{1/2} n_1^{1/2} |n - n(u)|_1^{-1} \text{ otherwise.} \end{cases}$$

Thus we get that the contribution to $S(u)$ of these error terms is $\ll G(hm)^{1/2} m_1^{1/2} M_1^{(k-1)/2} L^2$, which goes into the first error term of the formula.

We are now left with showing that the tail part in the summation formula, that is terms for $n > n_j$, are accounted for in the theorem. Here we make use of theorem 3.2 for the estimation of the exponential integral since for $n > n_j$ the integral has no saddle-points in the interval of integration. Here U will be the smoothing parameter with $J = 1$. The contribution of $I_n^{(j)}$ to S' equals

$$(2m)^{-1/2} \sum_{j=1}^{2} (-1)^{j-1} \sum_{n < n_j} a(n) n^{-(k-1)/2} e(-n\bar{h}/m) n^{-1/4} \times$$

$$\int_a^b \eta_1(x) x^{(k/2)-(3/4)} g(x) \left[1 + g_j\left(\frac{4\pi\sqrt{(nx)}}{m} \right) \right] e(p_{j,n}(x)) dx$$

where $\eta_1(x)$ is the weight function. Apply theorem 3.2 with $p_{j,n}(z)$ in place of $f(z)$ and $\mu \asymp m_1$. Note that the conditions of the theorem 3.3 are met if we choose $M = m^{-1} M_1^{-1/2} n^{1/2}$. The second term on the right hand side of the estimate given in theorem 3.3 is exponentially smalll and hence can be neglected because

$$M\mu \gg m^{-1} M_1^{-1/2} n^{1/2} m_1 \gg (n/n_1)^{1/2} F m_1^2 M_1^{-2} \gg (n/n_1)^{1/2} M_1^{\delta}.$$

The term corresponding to $U^{-1} G M^{-2}$ therein is

$$\ll G m^{3/2} M_1^{(k/2)+(1/4)} U^{-1} \sum_{n \gg n_1} a(n) n^{-(k-1)/2-(5/4)}$$

$$\ll G m^{3/2} M_1^{(k/2)+(1/4)} U^{-1} n_1^{-1/4} L$$

$$\ll G F^{-1/2} m M_1^{(k/2)+1} m_1^{-1/2} U^{-1} L$$

$$\ll G F h^{-3/2} m^{5/2} M_1^{(k-1)/2} m_1^{-1/2} U^{-1} L.$$

Thus proof of the theorem is complete up to the following error terms:

$$GUM_1^{(k-1)/2}L + F^{-1/2}Gh^{3/2}m^{-1/2}m_1^{1/2}M^{(k-1)/2+\varepsilon}U$$
$$+ GFh^{-3/2}m^{5/2}M_1^{(k-1)/2}m_1^{-1/2}U^{-1}L$$

The first and the last terms above coincide with the last term in the transformation formula if we choose $U = F^{1/2}h^{-3/4}m^{5/4}m_1^{-1/4}$. Then the second term above is

$$\ll Gh^{3/4}m^{3/4}m_1^{1/4}M_1 \ll G(hm)^{1/2}m_1^{1/2}M_1^{\delta}$$

which can be seen to go into the first O-term of the transformation formula. It only remains to be shown that the above choice of U satisfies our requirement: $M_1^{\delta} \ll U \leqslant 1/2\min(m_1, m_2)$. We have

$$Um_1^{-1} \ll U(M_1^{1+\delta}F^{-1/2})^{-1} \ll (hm)^{1/4}m_1^{-1/2}M_1^{-\delta} \ll M_1^{-\delta}.$$

For the other inequality

$$U \gg F^{1/2}h^{-3/4}m^{5/4}M_1^{-(1/4)+\delta} \gg M_1^{1/4+\delta}h^{-1/4}m^{3/4} \gg M_1^{\delta}.$$

This completes the proof of theorem.

In the case when $f(\tau) = \Sigma a(n)\sqrt{y}K_0(2\pi ny)\cos(2\pi nx)$ is an even Maass form of level N and character ε as in §2, we have the following transformation formula.

Theorem 3.6. *Under the notations and assumptions of theorem* 3.5 *with* $k = 1$ *we have*

$$\sum_{M_1 \leqslant n \leqslant M_2} a(n)g(n)e(f(n)) = A\sum_{j=1}(-1)^{j-1}\sum_{n<n_j} b(n)e(nh'/m) \times$$
$$n^{-1/4}x_{j,n}^{-1/4}g(x_{j,n})p''_{j,n}(x_{j,n})^{-1/2}e(p_{j,n}(x_{j,n}) + 1/8) +$$
$$O(G(|h|m)^{1/2}m_1^{1/2}L^2) +$$
$$O(F^{1/2}G|h|^{-3/4}m^{5/4}M_1^{1/10}m_1^{-1/4}L). \qquad \square$$

Proof. Note that the second error term above is slightly worse than the corresponding error term in theorem 3.5. This is because the Deligne's estimate which was used in theorem 3.5 has not been proved for non-holomorphic forms and the best known estimate is $a(n) \ll n^{1/5+\varepsilon}$. Let S, S' and $S(u)$ be as in the proof of theorem 3.5; further assume that $(m, N) = N$, the other case is similar. Thus

$$S(u) = \sum_{a \leqslant n \leqslant b} a(n)g(n)e(f(n))$$
$$= \sum_{a \leqslant n \leqslant b} a(n)[\cos(2\pi nr) + i\sin(2\pi nr)]g(n)e(f(n) - nr)$$
$$= S_1(u) + iS_2(u), \text{ say.}$$

We now apply summation formulae of §2 to $S_1(u)$ and $S_2(u)$ and proceed to evaluate the integrals as before.

$$
\begin{aligned}
S_1(u) &= \sum_{a\leqslant n\leqslant b} a(n)\cos(2\pi nr)g(n)e(f(n)-nr)\\
&= \frac{\varepsilon(\bar{h})\pi}{m}\sum_{n=1}^{\infty} a(n)\cos(-2\pi n\bar{h}/m)\times\\
&\qquad \int_a^b\left[2/\pi K_0\left(\frac{4\pi\sqrt{(nx)}}{m}\right)-Y_0\left(\frac{4\pi\sqrt{(nx)}}{m}\right)g(x)e(f(x)-rx)\mathrm{d}x\right.\\
&= \frac{\varepsilon(\bar{h})\pi}{m}\sum_{n=1}^{\infty} a(n)\cos(-2\pi n\bar{h}/m)[i_n+I_n],
\end{aligned}
$$

where

$$i_n=\pi/2\int_a^b K_0\left(\frac{4\pi\sqrt{(nx)}}{m}\right)g(x)e(f(x)-rx)\mathrm{d}x$$

and

$$I_n=-\int_a^b Y_0\left(\frac{4\pi\sqrt{(nx)}}{m}\right)g(x)e(f(x)-rx)\mathrm{d}x.$$

We first observe that the contribution from the integrals i_n is negligible. We have $\sqrt{(nM_1)}/m\gg\sqrt{nM_1^\delta}$ so that

$$
\begin{aligned}
m^{-1}\sum_{n=1}^{\infty} a(n)|i_n| &\ll m^{-1}G\sum_{n=1}^{\infty} a(n)\exp(-A\sqrt{nM_1^\delta})\\
&\ll G\exp(-AM_1^\delta).
\end{aligned}
$$

Write the integrals I_n in terms of the Hankel functions to get:

$$
\begin{aligned}
I_n &= i\int_a^b\left[H_0^{(1)}\left(\frac{4\pi\sqrt{(nx)}}{m}\right)-H_0^{(2)}\left(\frac{4\pi\sqrt{(nx)}}{m}\right)\right]g(x)e(f(x)-rx)\mathrm{d}x\\
&= I_n^{(1)}-I_n^{(2)},
\end{aligned}
$$

where

$$I_n^{(j)}=i\pi^{-1}m^{1/2}n^{-1/4}\int_a^b x^{-1/4}g(x)\left[1+g_j\left(\frac{4\pi\sqrt{(nx)}}{m}\right)\right]e(p_{j,n}(x))\mathrm{d}x.$$

Notice that this is same as the integral '$I_n^{(j)}$' in the proof of theorem 3.5 with $k=1$. Similarly for the sum $S_2(u)$ and putting these two terms together we get the transformation formula claimed in the theorem. Note also that the 'Rankin's trick' has been extended to the case of Maass forms to get the mean value estimate:

$$\sum_{n\leqslant X}|a(n)|^2=CX+O(X^{3/5+\varepsilon}).$$

We now proceed to give analogs of the above transformation formulae for smoothed exponential sums provided with weights of the type $\eta_J(n)$ of pp. 9. We get much better error terms but we will have to allow for certain weights to appear in the transformed sum as well.

Theorem 3.7. *Suppose that the assumptions of the theorem 3.5 are satisfied. Let* $U \gg F^{-1} M_1^{1+\delta} F^{1/2} r^{-1} M_1^{\delta}$, *and* J *be a fixed positive integer exceeding a certain bound. Write for* $j = 1, 2$

$$M'_j = M_j + (-1)^{j-1} JU = M(r) + (-1)^j m'_j,$$

and suppose that $m'_j m_j$. Let n_j *be as before and*

$$n'_j = (r - f'(M'_j))^2 m^2 M'_j.$$

Then defining the weights $\eta_J(x)$ *in the interval* $[M_1, M_2]$ *as in pp. 9 we have*

$$\begin{aligned}
&\sum_{M_1 \leqslant n \leqslant M_2} \eta_J(n) a(n) g(n) e(f(n)) \\
&\quad = A \sum_{j=1}^{2} (-1)^{j-1} \sum_{n < n_j} w_j(n) a'(n) e(nh'/m) \times \\
&\quad n^{-(k/2)+1/4} x_{j,n}^{(k/2)-(3/4)} g(x_{j,n}) \times \\
&\quad p''_{j,n}(x_{j,n})^{-1/2} e(p_{j,n}(x_{j,n}) + 1/8) + \\
&\quad O(F^{-1} G |h|^{3/2} m^{-1/2} M_1^{(k-1)/2} m_1^{1/2} UL).
\end{aligned}$$

where $w_j(n) = 1$ *for* $n < n_j$, *and* $w_j(n) \ll 1$ *for* $n < n'_j$; *further* $w_j(y)$ *and* $w'_j(y)$ *are piecewise continuous functions in* (n'_j, n_j) *with at most* $J - 1$ *discontinuities and* $w'_j(y) \ll (n_j - n'_j)^{-1}$ *for* $y \in (n'_j, n_j)$ *whenever* $w'_j(y)$ *exists.* □

The proof of this theorem is the same as that of theorem 3.5 but uses theorem 3.2 in place of theorem 3.1 for details see [6]. A similar theorem holds for the nonholomorphic case.

A particular case

We now want to specialise the transformation formula to the case of Dirichlet polynomials, that is to say, to

$$S(M_1, M_2) = \sum_{M_1 \leqslant n \leqslant M_2} a(n)^{-(k/2)-it}$$

when $M_1 < t/2\pi r < M_2$ with r satisfying the conditions of theorem 3.5 and where $a(n)$'s are Fourier coefficients of a cusp form of weight k. Such sums occur, for instance, while estimating the Dirichlet series (associated to cusp forms) on the critical line and studying their zeros on the critical line.

Here $g(z) = z^{-k/2}$, $f(z) = -(t/2\pi)\log z$ and $M(-r) = t/2\pi r$. The assumptions of the theorems 3.5 are satisfied (with $-r$ in place of r) if we choose $F = t$ and $G = M_1^{-k/2} r^{k/2} t^{-k/2}$. Then $n_j = h^2 m_j^2 M_j^{-1}$, $M_j = (t/2\pi r) + (-1)^j m_j$ and the function $p_{j,n}(x)$ takes the form

$$p_{j,n}(x) = -(t/2\pi)\log x + rx + (-1)^{j-1}(2\sqrt{(nx)}/\alpha - (k-1)/4 - 1/8)$$

where $\alpha = m$ if $(m, N) = N$ and $\alpha = m\sqrt{N}$ if $(m, N) = 1$. Assume for sake of simplicity that $(m, N) = N$; the other case is entirely similar. Thus $x_{j,n}$'s are the roots of the equation

$$p'_{j,n}(x) = -t/2\pi x + r + (-1)^{j-1}\sqrt{n}(m\sqrt{x})^{-1} = 0$$

or equivalently of the quadratic equation

$$x^2 - ((t/\pi r) + (n/h^2))x + (t/2\pi r)^2 = 0.$$

Therefore, since $x_{1,n} < x_{2,n}$, we have

$$x_{j,n} = \frac{t}{2\pi r} + \frac{n}{2h^2} + \frac{(-1)^j}{h^2}\left(\frac{n^2}{4} + \frac{hknt}{2\pi}\right)^{1/2}$$

and

$$(t/2\pi r)^2 x_{j,n}^{-1} = \frac{t}{2\pi r} + \frac{n}{2h^2} - \frac{(-1)^j}{h^2}\left(\frac{n^2}{4} + \frac{hknt}{2\pi}\right)^{1/2}.$$

To wirte the transformation formula here we need to calculate $2^{-1/2}m^{-1/2}x_{j,n}^{-3/4}p''_{j,n}(x_{j,n})^{-1/2}$ and $p_{j,n}(x_{j,n})$.

We have

$$p''_{j,n}(x_{j,n}) = t/2\pi x_{j,n}^2 + (-1)^j 2^{-1} n^{1/2} m^{-1} x_{j,n}^{-3/2}$$

So

$$\begin{aligned} 2mx_{j,n}^{3/2}p''_{j,n}(x_{j,n}) &= \pi^{-1}mtx_{j,n}^{-1/2} + (-1)^j n^{1/2} \\ &= (-1)^{j-1}h^2 n^{-1/2}(2(t/2\pi r)^2 x_{j,n}^{-1} - t/\pi r) + (-1)^j n^{1/2} \\ &= \pi^{-1/2}(2hkt)^{1/2}\left(1 + \frac{\pi n}{2hkt}\right)^{1/2}. \end{aligned}$$

Thus

$$2^{-1/2}m^{-1/2}x_{j,n}^{-3/4}p''_{j,n}(x_{j,n})^{-1/2} = \pi^{1/4}(2hkt)^{-1/4}\left(1 + \frac{\pi n}{2hkt}\right)^{-1/4}.$$

Calculation of $p_{j,n}(x_{j,n})$ is more delicate. We have

$$\begin{aligned} (2\pi r t^{-1} x_{j,n})^{(-1)^j} &= 1 + \frac{\pi n}{hkt} + \left(\left(\frac{\pi n}{hkt}\right)^2 + \frac{2\pi n}{hkt}\right)^{1/2} \\ &= \left(\left(\frac{\pi n}{2hkt}\right)^{1/2} + \left(1 + \frac{\pi n}{2hkt}\right)^{1/2}\right)^2 \end{aligned}$$

whence $\log(2\pi r t^{-1} x_{j,n}) = (-1)^j 2 \operatorname{arcsinh}((\pi n/2hkt)^{1/2})$. We also have, by $p'_{j,n}(x_{j,n}) = 0$,

$$\begin{aligned} 2\pi r x_{j,n} + 4\pi(-1)^{j-1}n^{1/2}x_{j,n}^{1/2}m^{-1} &= 2t - 2\pi r x_{j,n} \\ &= t - \frac{\pi n}{hk} + (-1)^{j-1}2t\left(\frac{\pi n}{2hkt} + \left(\frac{\pi n}{2hkt}\right)^2\right)^{1/2}. \end{aligned}$$

Thus

$$2\pi p_{j,n}(x_{j,n}) = (-1)^{j-1}\left(2t\phi\left(\frac{\pi n}{2hkt}\right) - \frac{\pi(k-1)}{2} - \frac{\pi}{4}\right) - t\log\left(\frac{t}{2\pi}\right) +$$

$$+ t\log r + t - \frac{\pi n}{hk},$$

where we have put $\phi(x) = \operatorname{arcsinh}(x^{1/2}) + (x + x^2)^{1/2}$. Thus we have

$$e(p_{j,n}(x_{j,n} + 1/8) = e(-n/2hk)\exp(i(-1)^{j-1}\left(2t\phi\left(\frac{\pi n}{2hkt}\right) - \frac{\pi(k-1)}{2} - \frac{\pi}{4}\right) \times$$
$$r^{it}\exp(i(t + \pi/4))(2\pi/t)^{it}$$

Thus finally we have:

$$\begin{aligned}S(M_1, M_2) &= \sum_{M_1 \leqslant n \leqslant M_2} a(n)n^{-(k/2)-it} = \\ &= \pi^{1/4}(2hkt)^{-1/4}\left(\frac{2\pi r}{t}\right)^{it}\exp(i(t + \pi/4)) \\ &\quad \times \sum_{j=1}^{2}\sum_{n<n_j} a(n)e\left(n\left(\frac{\bar{h}}{m} - \frac{1}{2hk}\right)\right) \\ &\quad n^{(1/4)-(k/2)}\left(1 + \frac{\pi n}{2hkt}\right)^{-1/} \\ &\quad \times \exp(i(-1)^{j-1}\left(2t\phi\left(\frac{\pi n}{2hkt} - \frac{\pi(k-1)}{2} - \frac{\pi}{4}\right)\right) \\ &\quad + O(hm_1^{1/2}t^{-1/2}L^2) + O(h^{-1/4}m^{3/4}m_1^{-1/4}L).\end{aligned}$$

The smoothed version in this case reads:

$$\begin{aligned}S(M_1, M_2) &= \sum_{M_1 \leqslant n \leqslant M_2} \eta_j(n)a(n)n^{-(k/2)-it} = \\ &= \pi^{1/4}(2hkt)^{-1/4}\left(\frac{2\pi r}{t}\right)^{it}\exp(i(t + \pi/4)) \\ &\quad \times \sum_{j=1}^{2}\sum_{n<n_j} a(n)e\left(n\left(\frac{\bar{h}}{m} - \frac{1}{2hk}\right)\right) \\ &\quad \times n^{(1/4)-(k/2)}\left(1 + \frac{\pi n}{2hkt}\right)^{-1/4}\exp(i(-1)^{j-1} \\ &\quad \times \left(2t\phi\left(\frac{\pi n}{2hkt} - \frac{\pi(k-1)}{2} - \frac{\pi}{4}\right)\right) + O(h^2m^{-1}m_1^{1/2}t^{-3/2}UL).\end{aligned}$$

It is advantageous to choose U as small as the condition

$$U \gg F^{-1/2}M_1^{1+\delta} \asymp F^{1/2}M_1^{1+\delta}r^{-1}, \quad \text{i.e. } U \asymp F^{1/2+\varepsilon}r^{-1}.$$

With this choice the above error term becomes $O(F^{-1/2+\varepsilon}G(|h|m)^{1/2}M_1^{(k-1)/2}m_1^{1/2})$.
As usual we have a similar formula for the non-holomorphic case.

Remark In the case of Dirichlet series coming from cusp-forms of higher level, $N \geqslant 1$, the point of interest is $t\sqrt{N}/2\pi$, and m_j, the length, satisfies: $t^{1/2+\delta} \ll m_1 \ll t$. We can manage to get the same transformation formulae taking $M(r) = t\sqrt{N}/2\pi$ where r is an approximation to $\sqrt{N}$ which satisfies:

$$|r - 1/\sqrt{N}| \ll t^{1/2}, r = h/m, m \ll t^{1/4}$$

with $(m, N) = 1$.

It can be verified that the order of n_j remains unaltered and so will other estimates which depended on $f'(M(r)) = r$. For example let us look at $|-r - f'(M_1)|$:

$$M_1 = t\sqrt{N}/2\pi - m_1 = t\sqrt{N}/2\pi(1 - 2\pi m_1/t\sqrt{N});$$

So

$$M_1^{-1} = 2\pi/t\sqrt{N}(1 - 2\pi m_1/t\sqrt{N})^{-1} \simeq 2\pi/t\sqrt{N}(1 + 2\pi m_1/t\sqrt{N}) \text{ as } m_1 \ll t^{1-\delta};$$

and $f'(M_1) = -t/2\pi M_1$. Thus

$$\begin{aligned}|-r - f'(M_1)| &= |r - 1/\sqrt{N}(1 + 2\pi m_1/t\sqrt{N})| \\ &= |r - 1/\sqrt{N} - 2\pi m_1/Nt| \asymp m_1 t^{-1} \\ &\quad \times (= m_1 F M_1^{-2}, \text{ as } F = M_1 = t).\end{aligned}$$

We will make use of this remark in our application to 'zeros on the critical line' in the next section.

4. Applications

In this section we give two applications of the transformation formula. The first application deals with the zeros on the critical line of the Dirichlet series $\phi(s)$ associated with holomorphic cusp forms and the second application deals with the order of $\phi(k\backslash 2 + it)$. In all these applications we use only Rankin's meanvalue estimate though in the case of holomorphic forms the estimate $a(n) \ll n^{(k-1)/2+\varepsilon}$ (Ramanujan – Petersson conjecture) is known due to Deligne. Thus these results go through in the case of Maass forms as well where the analogue of Rankin's estimate has been proved but Deligne's estimate has not yet been; the best result known here is $a(n) = O(n^{1/5+\varepsilon})$ due to Serre.

Zeros on the critical line

Consider the Dirichlet series $\phi(s) = \Sigma a(n) n^{-s}$ where $a(n)$'s are Fourier coefficients of a cusp form of weight k, level N and character ε; this series satisfies the following functional equation

$$(2\pi/\sqrt{N})^{-s}\Gamma(s)\phi(s) = C(2\pi/\sqrt{N})^{s-k}\Gamma(k-s)\psi(k-s),$$

where $|C| = 1$ (for a proof take $m = 1$ in theorem 2.1 of §2). If ε is a real character then $f \to f|_{H(N)}$ is an automorphism of $M(N, k, \varepsilon)$ and since it is an involution we can decompose $M(N, k, \varepsilon)$ further as $M^+(N, k, \varepsilon) + M^-(N, k, \varepsilon)$ where on $M^{\pm}(N, k, \varepsilon) H(N)$ acts by ± 1. Thus if $f \in M^{\pm}(N, k, \varepsilon)$ then $b(n) = \pm a(n)$ in the earlier notation. In this situation if we rewrite the functional equation as

$$\phi(s) = C'\Delta(s)\phi(k-s), \quad \Delta(s) = (2\pi/\sqrt{N})^{2s-k}\Gamma(k-s)/\Gamma(s), C' = \pm C$$

and further assume that $a(n)$'s are real we see that on the critical line $\Delta(s)$ has absolute value 1, $|\Delta((k/2)+it)|=1$. Therefore the function

$$Z_\phi(t)=[C'\Delta((k/2)+it))]^{-1/2}\phi((k/2)+it)$$

is a real function of t. We can now use this function to check whether $\phi(s)$ has any zeros on the critical line for t in an interval $[T-H,\ T+H]$ by comparing the integrals

$$\left|\int_{-H}^{H} Z_\phi(T+u)\mathrm{d}u\right| \text{ and } \int_{-H}^{H} |Z_\phi(T+u)|\mathrm{d}u$$

for if $\phi(s)$ does not vanish for t in the above interval then these two integrals should coincide.

Theorem 4.1. *Suppose that $a(n)$ is real for all n. Then for all $\varepsilon>0$ there exists a number $T_0=T_0(\varepsilon)$ such that for all $T\geqslant T_0$ the function $\phi(s)$ has a zero $(k/2+i\gamma)$ with $|T-\gamma|\leqslant T^{1/3+\varepsilon}$. A similar statement holds for the function $\phi(s,1/N)$.* □

Proof. We shall first prove the theorem for the function $\phi(s,1/N)$. Observe that for the Dirichlet series $\phi(s,1/N)$ also the corresponding function

$$Z_\phi(t)=[C\Delta(k/2)+it)]^{-1/2}\phi((k/2)+it,1/N)$$

is real by virtue of the functional equation proved in theorem 2.1. Also note that here ε need not be a real character and that the result is true for $\phi(s,h/m)$ where h is such that $h^2\equiv 1(\bmod m)$. Suppose that $\phi(k/2+it,1/N)$ does not vanish for t in the interval $[\tau-H,T+H]$. Then $Z_\phi(t)$ is of constant sign in the above interval. Let $H=T^{(1/3)+3\varepsilon}$ and consider the integral

$$I=\int_{-H}^{H} Z_\phi(T+u)\exp(-(u/H_0)^2)\mathrm{d}u,$$

where $H_0=T^{1/3+2\varepsilon}$

It is well known that

$$|I|=\int_{-H}^{H}|Z_\phi(T+u)|\exp(-(u/H_0)^2)\mathrm{d}u\gg\int_{-H_0}^{H_0}|Z_\phi(T+u)|\mathrm{d}u\gg H_0$$

See Theorem 3 in [1] for a proof.

We shall estimate I in a different way by making use of the following representation for $\phi(s,1/N)$ on the critical line:

Lemma 4.2 *Let $t\geqslant 2$ and $t^2\ll X\ll t^A$ where A is an arbitrary positive constant. Then we have, putting $a'(n)=a(n)e(1/N)$,*

$$\phi(k/2)+it,1/N)=\sum_{n\leqslant X}a'(n)n^{-(k/2)-it}+$$
$$+(\log 2)^{-1}\sum_{X<n\leqslant 2X}a'(n)\log(2X/n)n^{-(k/2)-it}+O(tX^{-1}).$$

□

Proof. The proof is standard (see for example [6]).

Take $X = T^3$ and let $K \in [T^{2/3-\varepsilon}, 2T^{2/3-\varepsilon}]$. We have

$$I = \sum_{\substack{n \leqslant T^3 \\ |n - TN/2\pi| > K}} a'(n) n^{-(k/2)-iT} C^{-1/2}$$

$$\times \int_{-H}^{H} \Delta(k/2 + i(T+u))^{-1/2} n^{-iu} \exp(-(u/H_0)^2) \mathrm{d}u$$

$$+ C^{-1/2} \int_{-H}^{H} \Delta(k/2 + i(T+u))^{-1/2} \left(\sum_{|n - TN/2\pi| \leqslant K} a'(n) n^{-(k/2)-i(T+u)} \right)$$

$$+ \exp(-(u/H_0)^2) \mathrm{d}u + (\log 2)^{-1} \sum_{T^3 < n \leqslant 2T^3} a'(n) \log(2T^3/n) n^{-(k/2)-iT} \times$$

$$\times C^{-1/2} \int_{-H}^{H} \Delta(k/2 + i(T+u))^{-1/2} n^{-iu} \exp(-(u/H_0)^2) \mathrm{d}u + O(1)$$

$$= I_1 + I_2 + I_3 + O(1).$$

We will now show that I_1 and I_3 are small. Let first $n > TN/2\pi + K$, and estimate the integral,

$$\int_{-H}^{H} \Delta(k/2 + i(T+u))^{-1/2} n^{-iu} \exp(-(u/H_0)^2) \mathrm{d}u,$$

by looking at the corresponding complex integral over the rectangular contour with vertices $\pm H$, $\pm H - iH_0$. By Sterling's formula we have (remember $\Delta(s) = (2\pi/N)^{2s-k} \Gamma(k-s)/\Gamma(s)$)

$$\Delta(k/2 + i(T+u))^{-1/2} n^{-iu} = \exp(i(T \log(TN/2\pi)$$
$$- T + u \log(TN/2\pi n) + O(1))).$$

On the vertical sides this is bounded and

$$\exp(-(u/H_0)^2) \ll \exp^{\varepsilon}(-T).$$

On the horizontal side in the lower half-plane $\exp(-(u/H_0)^2)$ is bounded and

$$\Delta(k/2 + i(T+u))^{-1/2} n^{-iu} \ll \exp\{-H_0 \log(2\pi n/NT)\} \ll \exp(-AT^{\varepsilon}).$$

For $n < TN/2\pi - K$ the corresponding integral can be estimated similarly by integrating in the upper half-plane. Thus I_1 and I_3 are $\ll 1$.

Coming to I_2 we have

$$I_2 \ll H \sup_{|T-t| \leqslant H} \left| \sum_{|n - TN/2\pi| \leqslant K} a'(n) n^{-(k/2)-it} \right|$$

$$\ll H \sup_{|T-t| \leqslant H} \left| \sum_{|n - tN/2\pi| \leqslant K} a'(n) n^{-(k/2)-it} \right| + O(HT^{-1/30+3\varepsilon/2})$$

The error was obtained by Rankin's estimate with error term:

$$\sum_{n\leqslant x} |a(n)|^2 = Ax^k + O(x^{k-2/5}).$$

We shall estimate the above sum by applying the transformation formula from § 3 with $r=1/N$ and $M_j = tN/2\pi + (-1)^j K$. Then $n_j \ll t^{1/3-2\varepsilon}$, and the above sum is $\ll T^{-3\varepsilon/2}$. Thus

$$|I| \ll H_0 T^{-\varepsilon/2}.$$

But this contradicts $|I| \gg H_0$ if T is sufficiently large. Hence the assertion.

Now, coming to the Dirichlet series $\phi(s)$ we have

$$\Delta(k/2+it)^{-1/2} n^{-iu} = \exp(i(T\log(T\sqrt{N}/2\pi) - T + u\log(T\sqrt{N}/2\pi n) + O(1).$$

Hence the sum which we will have to estimate will be over an interval around $T\sqrt{N}/2\pi$. Here we will have to use the remark made at the end of § 4. Because of the approximation of $\sqrt{N}$ by $r=h/m$ we will have to apply the smoothed version of the transformation formula. So instead of the integral I above we will start with its smoothed version I_J

$$I_J = \int_{-H}^{H} \eta_J(T+u) Z_\phi(T+u) \exp(-(u/H_0)^2)\,\mathrm{d}u.$$

As in the previous case we have $|I_J| \gg H_0$.

Proceeding as before but breaking the sum at $|n - T\sqrt{N}/2\pi| \leqslant K - v$ where $v = v_1 + v_2 + \cdots + v_J$ is the smoothing parameter, we get

$$I_J = I_1' + I_2' + I_3' + O(1),$$

where now

$$I_2' = \int_{-H}^{H} [C'\Delta((k/2)+i(T+u))^{-1/2}\eta_J(T+u)$$

$$\times \left(\sum_{|n-T\sqrt{N}/2\pi|\leqslant K-v} a(n) n^{-k/2-i(T+u)}\right)\exp(-(u/H_0)^2)\,\mathrm{d}u.$$

Thus

$$|I_2'| \ll H \sup_{|T-t|\leqslant H} \left| \sum_{|n-T\sqrt{N}/2\pi|\leqslant K-v} \eta_J(n) a(n) n^{-k/2-it} \right|.$$

Now estimating as in the previous case but now using the remark at the end of § 3 and smoothed version of the transformation formula we conclude that the above sum is $\ll T^{-3\varepsilon/2}$ and so I_2' is $\ll H_0 T^{-\varepsilon/2}$. The integrals I_1' and I_3' are estimated as before.

Estimation of 'long' sums and order of $\phi(k/2+it)$

Here we are concerned with exponential sums

$$\sum_{M\leqslant n\leqslant M'} a(n) g(n) e(f(n))$$

which are "long" in the sense that the length may be of the order of M itself. It is not practical to transform such sums directly as in § 3 because variations in $f'(x)$ might be too much in the interval $[M, M']$. It is advisable to first partition $[M, M']$ into segments such that $f'(x)$ practically remains a constant in each segment and then transform these short sums. But we need to assume that $f'(x)$ is approximately a power to be able to get some saving in the estimate. The precise result (theorem 4.6 in [6]) is as follows:

Theorem 4.3. *Let $2 \leqslant M < M' \leqslant 2M$ and let f be a holomorphic function in the domain $D = \{z \,|\, |z - x| < cM$ for some $x \in [M, M']\}$ where c is a positive constant. Suppose that $f(x)$ is real in $[M, M']$ and that either*

$$f(z) = Bz^{\alpha}(1 + O(F^{-1/3})), \quad z \in D$$

where $\alpha \neq 0$, 1 is a fixed real number and

$$F = |B| M^{\alpha}$$

or

$$f(z) = B \log z(1 + O(F^{-1/3})), \quad z \in D \text{ with } F = |B|.$$

Let $g \in C^1[M, M']$ and suppose that for $x \in [M, M']$

$$|g(x)| \ll G, \quad |g'(x)| \ll G'.$$

Assume further that $M^{3/4} \ll F \ll M^{3/2}$. Then

$$\left| \sum_{M \leqslant n \leqslant M'} a(n) n^{-(k-1)/2} g(n) e(f(n)) \right| \ll (G + MG') M^{1/2} F^{1/3+\varepsilon}$$

where $a(n)$'s are Fourier coefficients of a cusp form. □

We will not give a proof here since Jutila's proof for the full modular group case goes through except that a slight modification is required since (unlike in that case in our situation) we do not have transformation formulae for $M_1 < t/2\pi r < M_2$, where $r (= h/m)$ is a rational number, for all r; we need to assume that $(m, N) = 1$ or N (N is the level of the cusp form) to get a transformation formula. The required modification is as follows: Put $M_0 = F^{2/3+\delta}$ and let $K = (M/M_0)^{1/2}$. We may suppose that $M \geqslant M_0$ for, otherwise the assertion is trivial. Consider the Farey sequence of order K and drop all those fractions h/m with $(m, N) > 1$. Denote this set of fractions by $\mathbb{K}$. If $r = h/m$ and $r' = h'/m'$ are two consecutive fractions in $\mathbb{K}$ let $\rho = (h + h')/(m + m')$ be their 'mediant'. We have

$$\rho - r = (mh' - m'h)/m(m + m')$$

In the usual case we would have $\rho - r = 1/m(m + m')$; but order-wise both are same i.e. $1/mK$. Define the points $M(\rho)$ by $f'(M(\rho)) = \rho$ and break the given sum at points $M(\rho)$ lying in the interval $[M, M']$. The rest of the proof is as in [6].

COROLLARY 4.4

We have

$$|\phi(k/2 + it)| \ll (|t| + 1)^{1/3+\varepsilon}.$$ □

Proof. We have the following approximate functional equation for $\phi(s)$, for $0 \leqslant \sigma \leqslant k$ and $t \geqslant 10$:

$$\phi(s) = \sum_{n \leqslant x} a(n) n^{-s} + \psi(s) \sum_{n \leqslant y} b(n) n^{s-k} + O(x^{k/2-\sigma} \log t)$$

where

$$x, y \geqslant 1, \quad xy = (t\sqrt{N}/2\pi)^2 \text{ and } \psi(s) = (2\pi/\sqrt{N})^{2s-k}\Gamma(k-s)/\Gamma(s).$$

This reduces the proof of the corollary showing that for all (positive and negative) large values of t and for all M, M' with $1 \leqslant M < M' \leqslant t\sqrt{N}/2\pi$ and $M' \leqslant 2M$ we have

$$\left| \sum_{M \leqslant n \leqslant M'} a(n) n^{-k/2-it} \right| \ll t^{1/3+\varepsilon}.$$

This is precisely the estimate of the theorem 4.3 applied to this sum.

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Dirichlet L-function and power series for Hurwitz zeta function

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Abstract. For $0 < \alpha < 1$, let $\zeta(s,\alpha)$ be the Hurwitz zeta function and let $\zeta_1(s,\alpha) = \zeta(s,\alpha) - \alpha^{-s}$. For a fixed s, we develop $\zeta_1(s,\alpha)$ as a power series in α in the complex circle $|\alpha| < 1$. If

$$\sum_{\chi(\mathrm{mod}\, q)} L(s,\chi)\, L(s',\bar{\chi}) = \frac{\phi(q)}{q^{s+s'}} \sum_{k/q} \mu\left(\frac{q}{k}\right)\left(\sum_{a=1}^{k}\left(\frac{k}{a}\right)^{\mathrm{Re}\, s + \mathrm{Re}\, s'} + Q(s,s',k)\right),$$

we obtain an asymptotic expansion for $Q(k) = Q(s,s',k)$ using the power series for $\zeta_1(s,\alpha)$

Keywords. Hurwitz zeta function; Dirichlet L-function; power series.

For integer $q \geqslant 1$, let $\chi(\mathrm{mod}\, q)$ be a Dirichlet character and let $L(s,\chi)$ be the corresponding Dirichlet L-series, where $s = \sigma + it$. For $0 < \alpha \leqslant 1$, let $\zeta(s,\alpha)$ be Hurwitz zeta function, defined by

$$\zeta(s,\alpha) = \sum_{n \geqslant 0} (n+\alpha)^{-s} \text{ for } \mathrm{Re}\, s > 1;$$

and its analytic continuation. For $0 \leqslant \alpha \leqslant 1$, let

$$\zeta_1(s,\alpha) = \sum_{n \geqslant 1} (n+\alpha)^{-s} \text{ for } \mathrm{Re}(s) > 1;$$

and its analytic continuation.

Our aim is to prove the following two theorems. Theorem 1 deals with $\zeta_1(s,\alpha)$ as a function of complex variable α for fixed complex number s. Consequently, we shall get some identities involving Riemann zeta function. Using Theorem 1, we shall prove Theorem 2, which deals with

$$\sum_{\chi(\mathrm{mod}\, q)} L(s,\chi)\, L(s',\bar{\chi})$$

for fixed complex numbers $s = \sigma + it$ and $s' = \sigma' + it'$ with $0 < \sigma, \sigma' < 1$. We state our theorems now.

Theorem 1. *For a fixed complex number* $s \neq 1$, $\zeta_1(s,\alpha)$ *is an analytic function of* α *in the region* $|\alpha| < 1$ *of the complex plane, where it admits of the power series expansion*

$$\zeta_1(s,\alpha) = \sum_{n \geqslant 0} b_n(s)\alpha^n, \text{ where } b_0(s) = \zeta(s). \text{ For } n \geqslant 1,$$

$$b_n(s) = \frac{(-1)^n}{n!} s(s+1)(s+2)\ldots(s+n-1)\zeta(s+n).$$

Here $\zeta(s)$ is Riemann zeta function. The above expansion is valid also for $\alpha = 1$, when $0 < \operatorname{Re} s < 1$.

Theorem 2. *We have, for $s = \sigma + it$ and $s' = \sigma' + it'$ with $0 < \sigma, \sigma' < 1$;* $\sigma_0 = \max(\sigma, \sigma')$; *$t, t'$ real; and for integer $q \geqslant 1$,*

$$\frac{1}{\phi(q)} \sum_{\chi(\bmod q)} L(s,\chi) L(s',\bar{\chi}) = \sum_{\substack{a=1\\(a,q)=1}}^{q} a^{-s-s'} + q^{-s-s'} \sum_{k/q} \mu(q/k) Q(s,s',k),$$

where

$$Q(s,s',k) = \sum_{a=1}^{k} \left(\frac{k^s}{a^s} \zeta_1\left(s', \frac{a}{k}\right) + \frac{k^{s'}}{a^{s'}} \zeta_1\left(s, \frac{a}{k}\right) + \zeta_1\left(s, \frac{a}{k}\right) \zeta_1\left(s', \frac{a}{k}\right) \right)$$

$$= A(s,s')k + \sum_{n=0}^{N-1} (a_n(s,s')k^{s-n} + b_n(s,s')k^{s'-n} + c_n(s,s')k^{-n}) + O(k^{\sigma_0 - N})$$

for any $N \geqslant 1$. Here ϕ denotes Euler's function, and $A(s,s')$, $a_n(s,s')$, $b_n(s,s')$ and $c_n(s,s')$ are functions of s and s' alone (being independent of k); and are explicitly computable. The O-constant, which is dependent on s and s' alone is also explicitly computable.

Corollaries of Theorem 2.

COROLLARY 1.

For $s' = \bar{s}$, and $s = \sigma + it$ with $0 < \sigma < 1$,

$$\frac{1}{\phi(q)} \sum_{\chi(\bmod q)} |L(s,\chi)|^2 = \sum_{\substack{a=1\\(a,q)=1}}^{q} a^{-2\sigma} + q^{-2\sigma} \sum_{k/q} \mu(q/k) Q(s,\bar{s},k)$$

and hence for $s = \frac{1}{2} + it$,

$$\frac{q}{\phi(q)} \sum_{\chi(\bmod q)} |L(\tfrac{1}{2} + it, \chi)|^2 = q \sum_{\substack{a=1\\(a,q)=1}}^{q} \frac{1}{a}$$

$$+ \sum_{k/q} \mu(q/k) \left(A(s,\bar{s})k + \sum_{n=0}^{N-1} a_n(s,\bar{s})k^{s-n} + \sum_{n=0}^{N-1} b_n(s,\bar{s})k^{\bar{s}-n} \right.$$

$$\left. + \sum_{n=0}^{N-1} c_n(s,\bar{s})k^{-n} + O(k^{-N}) \right).$$

COROLLARY 2.

We have for prime p,

$$\sum_{\chi(\bmod p)} L(s,\chi) L(s',\bar{\chi}) + (p-1) \cdot p^{-s-s'} \cdot \zeta(s)\zeta(s')$$

$$=(p-1)\sum_{a=1}^{p} a^{-s-s'}+(p-1)p^{-s-s'}\left(A(s,s')p+\sum_{n=0}^{N-1} a_n(s,s')p^{s-n}\right.$$
$$\left.+\sum_{n=0}^{N-1} b_n(s,s')p^{s'-n}+\sum_{n=0}^{N-1} c_n(s,s')p^{-n}+O(p^{-N})\right)$$

Notation: In what follows, any function of s and s' will be termed 'scalar'. Thus, a scalar will be independent of q. Also $a_n = a_n(s,s')$, $b_n = b_n(s,s')$, $c_n = c_n(s,s'),\ldots$ shall denote sequence of scalars. Similarly $A = A(s,s')$, $B = B(s,s')$, $C = C(s,s'),\ldots$ shall denote scalars. The $\ll$ and O-constants shall depend on scalars only. p shall denote a prime number. $\phi(n)$, $\mu(n)$ will denote Euler's function and Moebius function respectively. $\zeta(s,1)=\zeta(s)$ shall denote Riemann zeta function. $\sum_{a=1}^{L}{}'$ shall denote summation $\sum_{a=1}^{L}$ with restriction $(a,q)=1$, for any L. We shall write

$$\psi_0(a)=a-[a]-\tfrac{1}{2}=-\sum_{n=1}^{\infty}\frac{\sin 2\pi na}{\pi n}.$$

ψ_1 shall denote the indefinite integral of $\psi_0(a)$. Thus

$$\psi_1(a)=\int^{a}\psi_0(t)\mathrm{d}t=2\sum_{n=1}^{\infty}\frac{\cos 2\pi na}{(2\pi n)^2}.$$

By induction $\psi_r(a)=$ indefinite integral of $\psi_{r-1}(a)$ i.e. $\int^{a}\psi_{r-1}(t)\mathrm{d}t$, for $r\geqslant 1$. Thus, for fixed r, $\psi_r(n)=C_r$ is a constant for integer n.

Paley [4], first showed that

$$\sum_{\substack{\chi(\operatorname{mod} q)\\ \chi\neq\chi_0}} L(s,\chi)\,L(s',\bar{\chi})=\phi(q)\,L(s+s',\chi_0)+A(s,s')\phi^2(q)q^{-s-s'}$$
$$+O(q^{1-\sigma+G}+q^{1-\sigma'+G'}),$$

where χ_0 denotes principal character $(\operatorname{mod} q)$, for $s+s'\neq 1$; and a corresponding expression, for $s+s'=1$. Here $\varepsilon,\varepsilon'>0$ are arbitrary and $A(s,s')$ is some function of s,s'. Heath–Brown [1] obtained our corollary 1 of Theorem 2 for $s=s'=\frac{1}{2}$. Motohashi [3] obtained Paley's result mentioned above for

$$\sum_{\chi(\operatorname{mod} p)} |L(\tfrac{1}{2}+it,\chi)|^2$$

with vastly improved error term viz. $O(p^{-1/2})$ for prime p. While, very recently Katsurada and Matsumoto [2] have obtained our corollary 1 of Theorem 2 for

$$\sum_{\chi(\operatorname{mod} q)} |L(s,\chi)|^2$$

in a slightly different form.

The method of our proof consists in the use of Euler summation formula (along with power series expansion of $\zeta_1(s,\alpha)$), which makes the statement of Theorem 2 almost obvious. We have used the fact that the expansion $\zeta_1(s,\alpha)=\sum_{n\geqslant 0} b_n(s)\alpha^n$ naturally separates 's part' and 'α part' of $\zeta_1(s,\alpha)$.

Next, we prove our Theorem 1.

Corollaries of Theorem 1

First, we state corollaries of Theorem 1.

COROLLARY

Putting $\alpha=1$, $-\frac{1}{2}$ and $\frac{1}{2}$ respectively in the statement of Theorem 1, *we get*

$$\text{I)}\quad 1-s\zeta(s+1)+\frac{s(s+1)}{2!}\zeta(s+2)-\frac{s(s+1)(s+2)}{3!}\zeta(s+3)+\cdots=0,\ \textit{for}\ 0<\operatorname{Re}s<1.$$

$$\text{II)}\quad (2^s-2)\zeta(s)=\sum_{n\geqslant 1}\frac{s(s+1)(s+2)\ldots(s+n-1)\zeta(s+n)}{n!\cdot 2^n}$$

$$\text{III)}\quad (2^s-2)\zeta(s)=2^s+\sum_{n\geqslant 1}\frac{(-1)^n s(s+1)\ldots(s+n-1)\zeta(s+n)}{n!\cdot 2^n}$$

Remarks on Theorem 1: 1) As $\sum_{n\geqslant 0} b_n(s)$ is convergent for $0<\operatorname{Re}s<1$, we see that the series $\zeta_1(s,\alpha)=\sum_{n\geqslant 0} b_n(s)\alpha^n$ is uniformly convergent on the closed interval $[0,1]$ of α. Hence multiplying $\zeta_1(s,\alpha)$ by a bounded Riemann-integrable function, we can integrate the consequent series term-by-term on $[0,1]$. 2) For positive integer k, we have,

$$\frac{\partial^k}{\partial\alpha^k}\zeta_1(s,\alpha)=(-1)^k s(s+1)(s+2)\ldots(s+k-1)\zeta_1(s+k,\alpha)$$

for any complex number s. 3) Considering real and imaginary parts of $\zeta_1(s,\alpha)$ separately as function of real variable α on the interval $[0,1]$, we get for any positive integer k, the following:

$$\begin{aligned}\zeta_1(s,\alpha)&=\zeta(s)-\alpha\cdot s\zeta(s+1)+\frac{\alpha^2}{2!}s(s+1)\zeta(s+2)+\cdots\\&\quad+\frac{(-1)^{k-1}}{(k-1)!}\alpha^{k-1}\cdot s(s+1)(s+2)\cdots(s+k-2)\zeta(s+k-1)\\&\quad+O\left(\frac{s(s+1)(s+2)\cdots(s+k-1)}{k!}\zeta(\sigma+k)\cdot\alpha^k\right)\quad\text{for }\operatorname{Re}s=\sigma>0,\end{aligned}$$

where the O-constant is absolute. 4) From

$$\zeta_1(s,\alpha)=\sum_{n\geqslant 0} b_n(s)\alpha^n,$$

by multiplying it with power series for

$$\zeta_1(s',\alpha)=\sum_{n\geqslant 0} b_n(s')\alpha^n,$$

we get

$$\zeta_1(s,\alpha)\zeta_1(s',\alpha)=\sum_{n\geqslant 0} d_n(s,s')\alpha^n,$$

where

$$d_n=d_n(s,s')=\sum_{i=0}^{n} b_i(s)b_{n-i}(s')$$

and the new series $\sum_{n\geqslant 0} d_n(s,s')\alpha^n$ also is uniformly convergent on the closed interval $[0,1]$ of α.

Proof of Theorem 1: Let $s=\sigma+it$ with $\sigma>1$ and α be real with $|\alpha|<1$. Then

$$\begin{aligned}\zeta_1(s,\alpha)&=\sum_{n\geqslant 1}(n+\alpha)^{-s}=\sum_{n\geqslant 1} n^{-s}\left(1+\frac{\alpha}{n}\right)^{-s}\\&=\sum_{n\geqslant 1} n^{-s}\left(1-s\cdot\frac{\alpha}{n}+\frac{s(s+1)\alpha^2}{2!n^2}-\frac{s(s+1)(s+2)}{3!}\frac{\alpha^3}{n^3}+\cdots\right)\\&=\sum_{n\geqslant 1} n^{-s}-\alpha s\sum_{n\geqslant 1} n^{-s-1}+\frac{s(s+1)\alpha^2}{2}\sum_{n\geqslant 1} n^{-s-2}-\cdots\\&=\zeta(s)-\alpha s\zeta(s+1)+\frac{s(s+1)}{2}\alpha^2\zeta(s+2)-\cdots\\&=\sum_{n\geqslant 0} b_n(s)\alpha^n,\text{ say.}\end{aligned}$$

Thus for $s=\sigma+it$, with $\sigma>1$; and real α, with $0<\alpha<1$, we have

$$\zeta_1(s,\alpha)=\sum_{n\geqslant 0} b_n(s)\alpha^n.$$

Next, we show the series on the right hand converges uniformly and absolutely on every compact subset of the complex plane with $|\alpha|<1$, for any complex number $s\neq 1$. For convenience, we take $s(\neq 1)$ to be a complex number such that $\mathrm{Re}(s)>0$. Now, the coefficients $b_n(s)$ are well-defined functions of s and for $n\geqslant 1$, if $\mathrm{Re}(s)>0$, then

$$\begin{aligned}|b_n(s)|&\leqslant\frac{|s|(|s|+1)(|s|+2)\cdots(|s|+n-1)\zeta(\sigma+n)}{n!}\\&\leqslant\frac{|s|(|s|+1)(|s|+2)\cdots(|s|+n-1)\zeta(\sigma+1)}{n!}.\end{aligned}$$

Let $|\alpha|\leqslant\beta<1$. Then, $|\sum_{n\geqq 1} b_n(s)\alpha^n|$

$$\leqslant\zeta(\sigma+1)\sum_{n\geqq 1}\frac{|s|(|s|+1)\cdots(|s|+n-1)\beta^n}{n!}\leqslant\zeta(\sigma+1)\{(1-\beta)^{-|s|}-1\}.$$

Thus by analytic continuation, for any fixed s with $\mathrm{Re}(s)>0$, $s\neq 1$, and for complex

α with $|\alpha| < 1$, $\zeta_1(s,\alpha)$ is an analytic function of α with the above power series expansion. This result can be extended for all complex s.

Next, we prove the validity of the above series for $\alpha = 1$. Let s be a complex number with $\mathrm{Re}(s) > 0$. Then we have to prove

$$\zeta_1(s,1) = \sum_{n\geqslant 0} b_n(s).$$

i.e. to prove

$$\zeta(s) - 1 = \zeta(s) + \sum_{n\geqslant 1} b_n(s).$$

This amounts to proving that

$$1 + \sum_{n=1}^{\infty} b_n(s) = 0.$$

We define

$$\zeta_0(s) = \zeta(s) - 1.$$

Consider

$$\begin{aligned} 1 &+ \sum_{n\geqslant 1} b_n(s) \\ &= 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} s(s+1)(s+2)\cdots(s+i-1)(1+\zeta_0(s+i)) \\ &= 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} s(s+1)(s+2)\cdots(s+i-1) \\ &\quad + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} s(s+1)(s+2)\cdots(s+i-1)\zeta_0(s+i) \\ &= 1 + A + B, \text{ say, provided both the sums exist.} \end{aligned}$$

Now

$$\begin{aligned} B &= \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} s(s+1)(s+2)\cdots(s+i-1)\zeta_0(s+i) \\ &= -s\sum_{n\geqslant 2} \frac{1}{n^{s+1}} + \frac{s(s+1)}{2!}\sum_{n\geqslant 2}\frac{1}{n^{s+2}} - \frac{s(s+1)(s+2)}{3!}\sum_{n\geqslant 2}\frac{1}{n^{s+3}} + \cdots \\ &= \sum_{n\geqslant 2} n^{-s}\left\{-\frac{s}{n} + \frac{s(s+1)}{2!}\frac{1}{n^2} - \frac{s(s+1)(s+2)}{3!}\frac{1}{n^3} + \cdots\right\} \\ &= \sum_{n\geqslant 2} n^{-s}\left\{\left(1+\frac{1}{n}\right)^{-s} - 1\right\} = \sum_{n\geqslant 2}((n+1)^{-s} - n^{-s}) = -2^{-s}. \end{aligned}$$

The inversion of the order of summation is justified for $\sigma = \mathrm{Re}(s) > 0$ by the convergence of

$$\sum_{n\geqslant 2} n^{-\sigma} \sum_{k=0}^{\infty} \frac{|s|(|s|+1)\cdots(|s|+k)}{(k+1)!n^{k+1}} = \sum_{n\geqslant 2} n^{-\sigma}\left(\left(1-\frac{1}{n}\right)^{-|s|} - 1\right)$$

Next,

$$1+A=1-s+\frac{s(s+1)}{2!}-\frac{s(s+1)(s+2)}{3!}+\cdots$$
$$=(1+1)^{-s}=2^{-s},\quad \text{for } 0<\operatorname{Re}s<1.$$

Thus we have proved

$$1+\sum_{n\geqslant 1} b_n(s)=2^{-s}-2^{-s}=0 \text{ for } 0<\operatorname{Re}(s)<1.$$

This completes the proof of our Theorem 1.

Proof of Theorem 2. For $s=\sigma+it$ and $s'=\sigma'+it'$ with $0<\sigma,\ \sigma'<1$, we have

$$\sum_{\chi(\operatorname{mod} q)} L(s,\chi)\,L(s',\bar{\chi})$$
$$=\sum_{\chi(\operatorname{mod} q)}\left(q^{-s}\sum_{\substack{a=1\\(a,q)=1}}^{q}\chi(a)\zeta\left(s,\frac{a}{q}\right)\right)\left(q^{-s'}\sum_{\substack{b=1\\(b,q)=1}}^{q}\bar{\chi}(b)\zeta\left(s',\frac{b}{q}\right)\right)$$
$$=q^{-s-s'}\sum_{\substack{a=1\\(a,q)=1}}^{q}\ \sum_{\substack{b=1\\(b,q)=1}}^{q}\zeta\left(s,\frac{a}{q}\right)\zeta\left(s',\frac{b}{q}\right)\sum_{\chi(\operatorname{mod} q)}\chi(a)\overline{\chi(b)}$$
$$=\phi(q)q^{-s-s'}\sum_{a=1}^{q}{}'\,\zeta\left(s,\frac{a}{q}\right)\zeta\left(s',\frac{a}{q}\right).$$

Next,

$$\sum_{a=1}^{q}{}'\,\zeta\left(s,\frac{a}{q}\right)\zeta\left(s',\frac{a}{q}\right)$$
$$=\sum_{a=1}^{q}{}'\left(\frac{q^s}{a^s}+\zeta_1\left(s,\frac{a}{q}\right)\right)\left(\frac{q^{s'}}{a^{s'}}+\zeta_1\left(s',\frac{a}{q}\right)\right)$$
$$=q^{s+s'}\sum_{a=1}^{q}{}'\frac{1}{a^{s+s'}}+q^{s}\sum_{a=1}^{q}{}'\frac{\zeta_1\left(s',\frac{a}{q}\right)}{a^s}+q^{s'}\sum_{a=1}^{q}{}'\frac{\zeta_1\left(s,\frac{a}{q}\right)}{a^{s'}}$$
$$+\sum_{a=1}^{q}{}'\,\zeta_1\left(s,\frac{a}{q}\right)\zeta_1\left(s',\frac{a}{q}\right).$$

Thus

$$\sum_{\chi(\operatorname{mod} q)} L(s,\chi)\,L(s',\bar{\chi})=\phi(q)\sum_{a=1}^{q}{}'\,a^{-s-s'}+\phi(q)q^{-s-s'}\cdot\sum_{k/q}\mu(q/k)Q(s,s',k),$$

where

$$Q(s,s',k)=\sum_{a=1}^{k}\zeta_1\left(s,\frac{a}{k}\right)\zeta_1\left(s',\frac{a}{k}\right)+\sum_{a=1}^{k}k^{s}\cdot\frac{\zeta_1\left(s',\frac{a}{k}\right)}{a^s}$$
$$+\sum_{a=1}^{k}\frac{k^{s'}\zeta_1\left(s,\frac{a}{k}\right)}{a^{s'}}$$

$$=\sum_{a=1}^{k} f_1\left(\frac{a}{k}\right)+\sum_{a=1}^{k} f_2\left(\frac{a}{k}\right)+\sum_{a=1}^{k} f_3\left(\frac{a}{k}\right), \quad \text{say.}$$

By Euler summation formula, for $1\leqslant i\leqslant 3$,

$$\sum_{a=1}^{k} f_i\left(\frac{a}{k}\right)=\int_1^k f_i\left(\frac{a}{k}\right)da+\int_1^k (a-[a]-\tfrac{1}{2})\left(\frac{d}{da}f_i\left(\frac{a}{k}\right)\right)da$$
$$+\left(\frac{f_i\left(\frac{1}{k}\right)+f_i(1)}{2}\right).$$

In view of finite power series expression for $\zeta_1\left(s,\frac{1}{k}\right)$ for any complex s, we find that $\dfrac{f_1\left(\frac{1}{k}\right)+f_1(1)}{2}$ has the required form. Thus

$$\frac{f_1\left(\frac{1}{k}\right)+f_1(1)}{2}=\sum_{n=0}^{N-1} a_n k^{-n}+O(k^{-N}) \quad (\text{for } N\geqslant 1),$$

where a_n's are some scalars. Similarly $\dfrac{f_2\left(\frac{1}{k}\right)+f_2(1)}{2}$ has the expression of the form

$$\sum_{n=0}^{N-1} a_n k^{s-n}+\sum_{n=0}^{N-1} b_n k^{-n}+O(k^{\sigma-N})$$

for some scalars a_n and b_n. Similarly, $\dfrac{f_3\left(\frac{1}{k}\right)+f_3(1)}{2}$ has the expression of the form

$$\sum_{n=0}^{N-1} a_n k^{s'-n}+\sum_{n=0}^{N-1} b_n k^{-n}+O(k^{\sigma'-N}).$$

Next, we evaluate

$$\int_1^k f_i\left(\frac{a}{k}\right)da \quad \text{for } 1\leqslant i\leqslant 3.$$

We have

$$\int_1^k \zeta_1\left(s,\frac{a}{k}\right)\zeta_1\left(s',\frac{a}{k}\right)da=\int_1^k f_1\left(\frac{a}{k}\right)da.$$

Noting

$$\zeta_1\left(s,\frac{a}{k}\right)=\sum_{n\geqslant 0} b_n(s)\left(\frac{a}{k}\right)^n; \quad \zeta_1\left(s',\frac{a}{k}\right)=\sum_{n\geqslant 0} b_n(s')\left(\frac{a}{k}\right)^n,$$

we have

$$\zeta_1\left(s,\frac{a}{k}\right)\zeta_1\left(s',\frac{a}{k}\right)=\sum_{n\geqslant 0} d_n\left(\frac{a}{k}\right)^n,$$

for sequence of scalars (d_n).

Thus

$$\int_1^k \zeta_1\left(s,\frac{a}{k}\right)\zeta_1\left(s',\frac{a}{k}\right)da=\sum_{n\geqslant 0} d_n\int_1^k\left(\frac{a}{k}\right)^n da$$

$$=\sum_{n\geqslant 0}\frac{d_n k^{-n}}{n+1}[a^{n+1}]_{a=1}^k=k\sum_{n\geqslant 0}\frac{d_n}{n+1}-\sum_{n\geqslant 0}\frac{d_n k^{-n}}{n+1}$$

$$=\left(\sum_{n\geqslant 0}\frac{d_n}{n+1}\right)k-\sum_{n\geqslant 0}^{N-1}\frac{d_n}{n+1}k^{-n}+O(k^{-N}).$$

Next,

$$\int_1^k f_2\left(\frac{a}{k}\right)da=\int_1^k\frac{k^s}{a^s}\zeta_1\left(s',\frac{a}{k}\right)da$$

$$=\sum_{n\geqslant 0}k^s\int_1^k a^{-s}\left(\sum_{n\geqslant 0}b_n(s')\left(\frac{a}{k}\right)^n\right)da$$

$$=\sum_{n\geqslant 0}b_n(s')k^{s-n}\cdot\int_1^k a^{n-s}da=\sum_{n\geqslant 0}b_n(s')k^{s-n}\left[\frac{a^{n-s+1}}{n-s+1}\right]_{a=1}^k$$

$$=k\sum_{n\geqslant 0}\frac{b_n(s')}{n-s-1}-\sum_{n\geqslant 0}\frac{b_n(s')}{n-s+1}k^{s-n}$$

$$=k\left(\sum_{n\geqslant 0}\frac{b_n(s')}{n-s-1}\right)-\sum_{n\geqslant 0}^{N-1}\frac{b_n(s')}{n-s+1}k^{s-n}+O(k^{\sigma-N}).$$

Similarly

$$\int_1^k\frac{k^{s'}}{a^{s'}}\zeta_1\left(s,\frac{a}{k}\right)da$$

$$=k\left(\sum_{n\geqslant 0}\frac{b_n(s)}{n-s'-1}\right)-\sum_{n\geqslant 0}^{N-1}\frac{b_n(s)}{n-s'-1}k^{s'-n}+O(k^{\sigma'-N}).$$

Thus

$$\int_1^k\left(\sum_{i=1}^3 f_i\left(\frac{a}{k}\right)\right)da$$

has the form

$$Ak+\sum_{n=0}^{N-1}b_n k^{s-n}+\sum_{n=0}^{N-1}c_n k^{s'-n}+O(k^{\sigma-N}+k^{\sigma'-N})$$

for some scalars A, b_n and c_n.

Next, we evaluate

$$\int_1^k\psi_0(a)\frac{d}{da}\left(f_i\left(\frac{a}{k}\right)\right)da,\quad\text{for } 1\leqslant i\leqslant 3.$$

Now

$$\int_1^k \psi_0(a)\frac{d}{da}f_1\left(\frac{a}{k}\right)da=\int_1^k \psi_0(a)\left(\frac{d}{da}\zeta_1\left(s,\frac{a}{k}\right)\zeta_1\left(s',\frac{a}{k}\right)\right)da.$$

On integration by parts N times, we get

$$\int_1^k \psi_0(a)\frac{d}{da}f_1\left(\frac{a}{k}\right)da=\sum_{i=1}^{N}\left[(-1)^{i-1}\psi_i(a)\frac{d^i}{da^i}\zeta_1\left(s,\frac{a}{k}\right)\zeta_1\left(s',\frac{a}{k}\right)\right]_{a=1}^{a=k}$$
$$+\int_1^k(-1)^N\psi_N(a)\frac{d^{N+1}}{da^{N+1}}\left(\zeta_1\left(s,\frac{a}{k}\right)\zeta_1\left(s',\frac{a}{k}\right)\right)da,$$

where $[f(a)]_{a=1}^{a=k}$ denotes $f(k)-f(1)$ for any function f. Noting, $\frac{d^i}{da^i}\left(\zeta_1\left(s,\frac{a}{k}\right)\right)$ is a scalar multiple of $k^{-i}\zeta_1\left(s+i,\frac{a}{k}\right)$ for any complex s; and also noting $\psi_i(n)=\psi_i=\text{constant}$ for any integer n; and expanding $\zeta_1\left(s,\frac{1}{k}\right)$ as a finite power series in $\frac{1}{k}$ for any s, we find that the integrated terms have the form

$$\sum_{n=1}^{N-1} b_n k^{-n}+O(k^{-N})$$

and the integral

$$\int_1^k da\psi_N(a)\frac{d^{N+1}}{da^{N+1}}\zeta_1\left(s,\frac{a}{k}\right)\zeta_1\left(s',\frac{a}{k}\right)$$
$$\ll k^{-N-1}\int_1^k|\psi_N(a)|da\ll k^{-N-1}\int_1^k da\ll k^{-N}.$$

Next, we consider

$$\int_1^k \psi_0(a)\frac{d}{da}f_2\left(\frac{a}{k}\right)da.$$

We can treat

$$\int_1^k \psi_0(a)\frac{d}{da}f_3\left(\frac{a}{k}\right)da$$

similarly.

Now

$$\int_1^k \psi_0(a)\frac{d}{da}f_2\left(\frac{a}{k}\right)da=k^{s'}\cdot\int_1^k da\psi_0(a)\frac{d}{da}\left(\zeta_1\left(s,\frac{a}{k}\right)a^{-s'}\right).$$

On integration by parts N times, we get

$$\int_1^k \psi_0(a)\frac{d}{da}\left(f_2\left(\frac{a}{k}\right)\right)da = k^{s'}\cdot\sum_{i=1}^{N}(-1)^{i-1}\left[\psi_i(a)\frac{d^i}{da^i}\left(\zeta_1\left(s,\frac{a}{k}\right)a^{-s'}\right)\right]_{a=1}^{k}$$

$$-k^{s'}\cdot\int_1^k(-1)^{N-1}\psi_N(a)\cdot\frac{d^{N+1}}{da^{N+1}}\left(\zeta_1\left(s,\frac{a}{k}\right)\cdot a^{-s'}\right)da.$$

Considering finite power series for $\zeta_1\left(s,\frac{1}{k}\right)$ for any s, as before, we find the integrated terms have the form

$$\sum_{n=0}^{N-1} a_n k^{s'-n} + \sum_{n=1}^{N-1} b_n k^{-n} + O(k^{-N})$$

for some scalars (a_n) and (b_n).

Now

$$k^{s'}\int_1^k \psi_N(a)\frac{d^{N+1}}{da^{N+1}}\left(\zeta_1\left(s,\frac{a}{k}\right)\cdot a^{-s'}\right)da = k^{s'}\int_1^k \psi_N(a)\left(\frac{d^{N+1}}{da^{N+1}}\zeta_1\left(s,\frac{a}{k}\right)\right)a^{-s'}\,da$$

$$+\sum_{j=0}^{N} a(j)k^{s'-j}\int_1^k \psi_N(a)\zeta_1\left(s+j,\frac{a}{k}\right)\cdot a^{-s'-1-r}\,da, \text{ where } r+j=N.$$

Here $a(j)$ are some scalars. Note that

$$k^{s'}\int_1^k \psi_N(a)\cdot\left(\frac{d^{N+1}}{da^{N+1}}\zeta_1\left(s,\frac{a}{k}\right)\right)a^{-s'}\,da$$

$$\ll k^{\sigma'}\int_1^k |\psi_N(a)|\cdot k^{-(N+1)}\left|\zeta_1\left(s+N+1,\frac{a}{k}\right)\right|a^{-\sigma'}\,da$$

$$\ll k^{\sigma'-(N+1)}\int_1^k a^{-\sigma'}\,da \ll k^{-N}.$$

On substitution

$$\zeta_1\left(s+j,\frac{a}{k}\right) = \sum_{i=0}^{r-1} b_i(s+j)\left(\frac{a}{k}\right)^i + O\left(\left(\frac{a}{k}\right)^r\right),$$

we find that

$$k^{s'-j}\int_1^k \psi_N(a)\zeta_1\left(s+j,\frac{a}{k}\right)a^{-s'-1-r}\,da = k^{s'-j}\int_1^k \frac{da\cdot\psi_N(a)}{a^{s'+1+r}}$$

$$\times\left(b_0(s+j)+b_1(s+j)\frac{a}{k}+b_2(s+j)\frac{a^2}{k^2}\right.$$

$$\left.+\cdots+b_{r-1}(s+j)\left(\frac{a}{k}\right)^{r-1}+O\left(\frac{a}{k}\right)^r\right).$$

The

$$O-\text{term}=k^{s'-j-r}\int_1^k \frac{\psi_N(a)}{a^{s'+1}}da \ll k^{\sigma'-N},$$

as $r+j=N$. For $1\leqslant i\leqslant r-1$, the ith term $=k^{s'-i-j}\cdot\int_1^k \frac{\psi_N(a)b_i(s+j)}{a^{s'+1+r-i}}da.$

Hence,

$$\begin{aligned}
&k^{s'-i-j}\int_1^k \frac{\psi_N(a)}{a^{s'+1+r-i}}da\\
&=k^{s'-i-j}\left(\int_1^\infty \frac{\psi_N(a)da}{a^{s'+1+(r-i)}}-\int_k^\infty \frac{\psi_N(a)da}{a^{s'+1+(r-i)}}\right)\\
&=c_{ij}k^{s'-i-j}-k^{s'-i-j}\int_k^\infty \frac{\psi_N(a)da}{a^{s'+1+(r-i)}},\quad \text{say.}\\
&=c_{ij}k^{s'-i-j}+O(k^{-r-j})=c_{ij}k^{s'-i-j}+O(k^{-N}),\quad \text{as } r+j=N
\end{aligned}$$

Thus,

$$\begin{aligned}
&k^{s'-j}\int_1^k \zeta_1\left(s+j,\frac{a}{k}\right)\cdot a^{-s'-1-r}\psi_N(a)da\\
&=\sum_{i=0}^{r-1}\alpha_{ij}k^{s'-i-j}+O(k^{\sigma'-N}),
\end{aligned}$$

where α_{ij} are some scalars, $0\leqslant j\leqslant N+1$ and $r+j=N$. Taking together the terms corresponding to all j, we find that

$$k^{s'}\cdot\int_1^k \psi_N(a)\frac{d^{N+1}}{da^{N+1}}\left(\zeta_1\left(s,\frac{a}{k}\right)a^{-s'}\right)da$$

has the form

$$\sum_{n=0}^{N-1}a_nk^{s'-n}+O(k^{\sigma'-N})$$

and hence consequently,

$$k^{s'}\cdot\int_1^k \psi_0(a)\frac{d}{da}\left(a^{-s'}\cdot\zeta_1\left(s,\frac{a}{k}\right)\right)da$$

has the form $\sum_{n=0}^{N-1}a_nk^{s'-n}+\sum_{n=0}^{N-1}b_nk^{-n}+O(k^{\sigma'-N})$ for some scalars a_n, b_n. Thus, $\sum_{a=1}^{k}f_2\left(\frac{a}{k}\right)$ has the form

$$Ak+\sum_{n=0}^{N-1}a_nk^{s'-n}+\sum_{n=0}^{N-1}b_nk^{-n}+O(k^{\sigma'-N}).$$

Similarly, we find that $\sum_{a=1}^{k} f_3\left(\frac{a}{k}\right)$ has the form

$$Ak + \sum_{n=0}^{N-1} a_n k^{s-n} + \sum_{n=0}^{N-1} b_n k^{-n} + O(k^{\sigma-n}).$$

This completes the proof of Theorem 2.

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Determinants of parabolic bundles on Riemann surfaces*

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Abstract. Let X be a compact Riemann surface and $M_s^p(X)$ the moduli space of stable parabolic vector bundles with fixed rank, degree, rational weights and multiplicities. There is a natural Kähler metric on $M_s^p(X)$. We obtain a natural metrized holomorphic line bundle on $M_s^p(X)$ whose Chern form equals mr times the Kähler form, where m is the common denominator of the weights and r the rank.

Keywords. Parabolic bundles; Riemann surfaces; Kähler metric; Chern forms; π-bundles; Hermitian line bundles; parabolic determinant bundles.

1. Introduction

In a foundational paper [18] in 1985, Quillen studied determinants of $\bar{\partial}$-operators over a compact Riemann surface. Suppose X is a compact Riemann surface whose genus g is at least 2. If E is a C^∞ vector bundle of rank r and degree d on X, the space $\mathscr{A}$ of all holomorphic structures (or $\bar{\partial}$-operators) in E is a complex affine space. For each $D \in \mathscr{A}$, the object

$$\mathscr{L}_D = \lambda(\ker D)^* \otimes \lambda(\operatorname{coker} D),$$

where λ denotes the top exterior power, is a well-defined line and, as D varies in $\mathscr{A}$ these lines form a holomorphic line bundle $\mathscr{L}$ on $\mathscr{A}$, which is called the determinant line bundle. Using the theory of analytic torsion, Quillen defined a smooth Hermitian metric in the determinant bundle and proved that its curvature equals a natural Kähler form on $\mathscr{A}$.

The theorem of Quillen has an immediate application to moduli spaces. The collection $\mathscr{A}_s$ of stable holomorphic structures is an open subset of $\mathscr{A}$ and the complex gauge group $\mathscr{P}\mathscr{G} = \mathrm{Aut}(E)/\mathbb{C}^*$ acts freely on $\mathscr{A}_s$. The orbit space of this action is the moduli space M_s of stable bundles of rank r and degree d on X. When $d = r(g-1)$, the action of $\mathscr{P}\mathscr{G}$ on $\mathscr{A}_s$ lifts to an action on $\mathscr{L}$ and, hence, the determinant $\mathscr{L}$ descends to a holomorphic line bundle L over M_s. The Quillen metric in $\mathscr{L}$ induces a metric in the bundle L on M_s. It can be shown, as a corollary to Quillen's theorem, that the curvature of this metric in L is a naturally defined Kähler form on the moduli space M_s. In particular, the metric in L is positive. In the above argument the restriction $d = r(g-1)$ is purely technical and can be bypassed. Thus a modification of the above construction yields a positive line L on M_s for all arbitrary r and d. When r and d are coprime, M_s is known to be compact, so Kodaira's embedding theorem implies that the positive line bundle L is ample on M_s.

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* After this paper was written we came across an announcement of similar results by Witten [22].

There is another interesting interpretation of the above corollary to Quillen's theorem. A well known result of Narasimhan and Seshadri [16] shows that the moduli space is isomorphic to a certain variety R of irreducible projective unitary representations of the fundamental group $\pi_1(X)$. For each $\rho:\pi_1(X) \to PU(r)$, in R let $\mathrm{ad}\rho:\pi_1(X) \to \mathrm{Aut}(\mathfrak{g})$ denote the composition with ρ of the adjoint representation of $PU(r)$ on $\mathfrak{g} = \mathrm{Lie}\, PU(r)$ and let $W(\mathrm{ad}\rho)$ denote the local system of real vector spaces on X defined by ad ρ. Then the real tangent space to R at ρ is isomorphic to the de Rham group $H^1(X, W(\mathrm{ad}\rho))$.

Define

$$\Omega_\rho : H^1(X, W(\mathrm{ad}\rho)) \times H^1(X, W(\mathrm{ad}\rho)) \to \mathbb{R}$$

by

$$\Omega_\rho(\alpha, \beta) = \int_X \mathrm{tr}(\alpha \wedge \beta).$$

As ρ varies, the family Ω_ρ gives a 2-form Ω on R. This 2-form Ω is symplectic structure on R. One can ask wether Ω represents an integral cohomology class in $H^2(R, \mathbb{R})$. Yes is the answer to this question: under the isomorphism $R \simeq M_s$, Ω corresponds to the Kähler form on M_s which, in turn, equals the Chern form of the metrized line bundle L on M_s. Since all Chern classes are integral cohomology classes, Quillen's theorem implies that the symplectic form Ω on R is integral.

In this paper, we extend these results to the case of stable parabolic bundles on Riemann surfaces. Parabolic bundles are vector bundles over marked Riemann surfaces with weighted flags in the fibres at the marked points. There are two ways to study parabolic bundles: one way is to approach parabolic bundles directly; an alternative approach is through π-bundles. A π-bundle is an equivariant vector bundle over a Riemann surface provided with an action of a finite group π. The above results easily go through for π-bundles and define a metrized π-determinant. The more difficult part is to interpret this π-determinant in terms of the parabolic structure. Reserving the relevant definitions till later, we just state the main results here.

Theorem 1.1. *Let Y be a compact Riemann surface and π a finite group acting effectively on Y. Denote by $M_s^\pi(Y)$ the moduli space of stable π-bundles of rank r and fixed local type on Y. Then there exists on $M_s^\pi(Y)$ a metrized line bundle L^π whose Chern form equals r times the natural Kähler form Θ^π.*

Using a formalism of Deligne, Beilinson and Manin [3, 5] we interpret the above theorem in terms of parabolic structures as follows.

Theorem 1.2. *Let X be a compact Riemann surface and J a finite subset of X. Denote by $M_s^p(X)$ the moduli space of stable parabolic bundles of rank r and degree d with parabolic structure concentrated on J and having fixed rational weights, fixed multiplicities and fixed top exterior power. Then there exists on $M_s^p(X)$ a metrized line bundle L^p whose Chern form equals mr times the natural Kähler form, where m is the common denominator of all the weights.*

Now we sketch the outline of the paper. In §2 we recall the basic notions about parabolic bundles and π-bundles, and display the relation between them. The determinants of π-bundles form the topic of §4 and here Theorem 1.1 is proved. The π-determinant is expressed in terms of parabolic structures in §4. The proof of Theorem 1.2 is contained in § 5 and, finally, in § 6 we give an application of our results.

2. Parabolic bundles and π-bundles

In this section we define parabolic bundles and π-bundles and describe how the two notions are related to each other. The two basic references for this section are Seshadri [20] and, Mehta and Seshadri [13].

Let X be a compact Riemann surface and E a holomorphic vector bundle over X.

DEFINITION 2.1

A quasi-parabolic structure on E at a point $x \in X$ is a strictly decreasing flag

$$E_x = F^1 E_x \supsetneq F^2 E_x \ldots\ldots \supsetneq F^k E_x \supsetneq F_x^{k+1} E = 0$$

of linear sub-spaces in E_x. We define

$$r_j = \dim F^j E_x - \dim F^{j+1} E_x.$$

The integer k is called the length of the flag and the sequence $(r_1, \ldots, r_k)$ is called the type of the flag. A parabolic structure in E at x is, by definition, a quasi-parabolic structure at x as above, together with a sequence of real numbers $0 \leqslant \alpha_1 < \ldots < \alpha_k < 1$. The α_j are called the weights and we set

$$d_x(E) = \sum_{j=1}^{k} r_j \alpha_j.$$

DEFINITION 2.2

Fix a finite set I of points in X. We say that E is a parabolic bundle with parabolic structure on I if we are given a parabolic structure in E at each point $x \in I$. The parabolic degree, $p\deg(E)$, is defined by

$$p\deg(E) = \deg(E) + \sum_{x \in I} d_x(E),$$

where $\deg(E)$ denotes the topological degree of E, and we put

$$p\mu(E) = p\deg(E)/\mathrm{rank}(E).$$

The points in I are called the parabolic points.

DEFINITION 2.3

Let E and F be parabolic bundles of the same type over X (i.e. E and F have the same parabolic points, the same types of flags, same weights etc.). A morphism from

E to F is a homomorphism of vector bundles $f: E \to F$, which preserves the flags at the parabolic points.

DEFINITION 2.4

Let E be parabolic bundle with parabolic locus I. A parabolic subbundle of E is a parabolic bundle F with parabolic locus I, such that:
(a) F is a subbundle of E; and
(b) for each point $x \in I$ and for every $j \in \{1, \ldots, k_x\}$, we have $\alpha_{x,j}(F) = \alpha_{x,i}(E)$, where i is the largest integer such that $F^j F_x \subseteq F^i E_x$ and k_x is the length of the flag in F_x.

DEFINITION 2.5

A parabolic bundle E is said to be semistable if for every proper subbundle F of E, we have

$$p\mu(F) \leqslant p\mu(E). \tag{2.1}$$

In case strict inequality holds in (2.1), for all F, we say that E is stable.

In [13], Mehta and Seshadri construct the moduli space $M^p(X)$ of semistable parabolic bundles over X with fixed rank, degree and parabolic type, and show that $M^p(X)$ is a projective variety. They also prove that the set $M_s^p(X)$ of stable parabolic bundles is an open smooth subset of $M^p(X)$.

We now proceed to define π-bundles. Consider a compact Riemann surface Y and let π be a finite group acting holomorphically and effectively on the left of Y. Then $X = \pi \backslash Y$ has a natural structure of a Riemann surface such that the canonical projection $p: Y \to X$ is a ramified covering. The ramification points of p in Y are exactly those points y where the isotropy subgroup π_y of y in π is non-trivial. Note that π_y is a cyclic subgroup of π for every $y \in Y$.

DEFINITION 2.6

A π-bundle over Y is a holomorphic vector bundle E over Y such that:
(a) the bundle projection $p: E \to Y$ is π-equivariant;
(b) if $y \in Y$ and $\gamma \in \pi$, the map $E_y \to E_{\gamma \cdot y}$ given by $E \mapsto \gamma \cdot \xi$, is a linear isomorphism.

DEFINITION 2.7

A π-subbundle of a π-bundle E is a subbundle F of E which is invariant under the action of π on E.

Notation 2.8. If E is a vector bundle over Y (which need not be a π-bundle), we denote

$$\mu(V) = \frac{\text{degree}(V)}{\text{rank}(V)}.$$

DEFINITION 2.9

A π-bundle E is said to be semistable if for every proper subbundle F of E (note that we do not assume that F is a π-subbundle of E), we have

$$\mu(F) \leqslant \mu(E). \tag{2.2}$$

We say that a π-bundle E is stable if for every proper π-subbundle F of E, we have a strict inequality

$$\mu(F) < \mu(E). \tag{2.3}$$

Remark 2.10 Given a π-bundle E of rank r over Y and a point $y \in Y$, there exists a π_y-invariant neighbourhood U of y in Y such that $E|_U$ is defined by a representation $\rho: \pi_y \to GL(r)$, and ρ is unique up to equivalence (cf. Seshadri [20], Proposition 2 of §2). The equivalence class of the representation ρ is called the local type of E at y and is completely determined by: (a) an integer k such that $1 \leqslant k \leqslant r$; (b) a sequence of integers $0 \leqslant l_1 \leqslant l_2 \leqslant \ldots \leqslant l_k \leqslant n-1$, where $n = |\pi_y|$; and, (c) positive integers $r_1, r_2, \ldots, r_k$ such that $r_1 + \ldots, + r_k = r$. In fact, given the above data, we can define ρ as the block matrix

$$\rho(\zeta) = \begin{bmatrix} \exp(2\pi i\alpha_1)\cdot I_1 & & 0 \\ & \ddots & \\ 0 & & \exp(2\pi i\alpha_k)\cdot I_k \end{bmatrix}$$

where ζ is a generator of π_y, $i^2 = -1$, $\alpha_j = l_j/n$ and I_j is the identity matrix of order r_j.

In [20] Seshadri constructs the moduli space $M^\pi(Y)$ of semistable π-bundles of fixed rank, degree and local type over Y and shows that $M^\pi(Y)$ is a projective variety. It is proved by him that the set $M_s^\pi(Y)$ of stable π-bundles is a smooth open subset of $M^\pi(Y)$.

We now describe the connection between π-bundles on Y and parabolic bundles on $X = \pi \backslash Y$. Let E be a π-bundle over Y. Consider the invariant direct image $p_*^\pi \mathcal{O}(E)$ of $\mathcal{O}(E)$. This is the $\mathcal{O}_X$-module defined on each open set $U \subseteq X$ by

$$p_*^\pi \mathcal{O}(E) = \{\pi\text{-invariant sections of } E \text{ on } p^{-1}(U)\}.$$

Since p is a proper map, $p_*^\pi \mathcal{O}(E)$ is a coherent $\mathcal{O}_X$-module and is clearly torsion-free. Since $\dim(X) = 1$, it follows that $p_*^\pi \mathcal{O}(E)$ is a locally free $\mathcal{O}_X$-module of the same rank as E, and we denote the corresponding vector bundle on X by E^π.

Denote by J the ramification set of p in X. For each $x \in J$ choose a point $y(x) \in p^{-1}(x)$ and let $n_x = |\pi_{y(x)}|$. Let E be a π-bundle of rank r over Y. Suppose that the local type (cf. Remark 2.10) of Y is given by the integers $0 \leqslant l_{x,1} \leqslant l_{x,2} \leqslant \ldots \leqslant l_{x,k_x} \leqslant n_x - 1$ and $(r_{x,1}, \ldots, r_{x,k_x})$. We then have the following result for which we refer to Bhosle [4] and Mehta–Seshadri [13].

PROPOSITION 2.11

There exists a natural parabolic structure in the bundle E^π on X with parabolic set $I = p(J)$. For each $x \in I$, the type of the flag in E_x^π is $(r_{x,1}, \ldots, r_{x,k_x})$ and the weights at x are $(\alpha_{x,1}, \ldots, \alpha_{x,k_x})$, where $\alpha_{x,j} = l_{x,j}/n_x$. Conversely, given a parabolic bundle F on X of the above parabolic type, there exists on Y a π-bundle E, unique up to isomorphism such that $E^\pi \simeq F$ as parabolic bundles. The correspondence $E \mapsto E^\pi$ takes stable π-bundles on Y to stable parabolic bundles on X and semistable π-bundles to semistable parabolic bundles. The resulting bijection between the semistable moduli spaces $M^\pi(Y)$ and $M^P(X)$ is an isomorphism of varieties.

3. The determinant bundle on $M_s^\pi(Y)$

Consider a compact Riemann surface Y and let π be a finite group acting holomorphically and effectively on Y. Let μ be a Hermitian metric on Y which is invariant under the action of π. Denote by $A^{p,q}$ the space of all C^∞ (p,q)-forms on Y and set $A^r = \Sigma_{p+q=r} A^{p,q}$. Let $\omega \in A^{1,1}$ be the fundamental $(1,1)$-form of the metric μ and assume that μ is normalized so that $\int_Y \omega = 1$.

Fix a C^∞ π-vector bundle E of rank r and degree d over Y. Let h be a π-invariant Hermitian metric in E. Denote by $A^{p,q}(E)$ the space of C^∞ (p,q)-forms on Y with values in E and let $A^r(E) = \Sigma_{p+q=r} A^{p,q}(E)$. There is a natural linear action of π on each of the spaces $A^{p,q}$ and $A^{p,q}(E)$. We denote the corresponding subspaces of π-invariant forms by $(A^{p,q})^\pi$ and $(A^{p,q}(E))^\pi$.

DEFINITION 3.1

A holomorphic structure in E is a $\mathbb{C}$-linear map $D: A^0(E) \to A^{0,1}(E)$ which satisfies the Leibnitz identity

$$D(fs) = f.D(s) + \bar{\partial} f \otimes s$$

for all $f \in A^0$ and $s \in A^0(E)$.

If D_1 and D_2 are holomorphic structures, the difference

$$\alpha = D_1 - D_2 : A^0(E) \to A^{0,1}(E)$$

is an A^0-linear map, i.e., $\alpha \in A^{0,1}(\text{End } E)$. Conversely, if D_1 is a holomorphic structure and if $\alpha \in A^{0,1}(\text{End } E)$, the sum $D_2 = D_1 + \alpha$ is again a holomorphic structure. Thus the set $\mathscr{A}$ of all holomorphic structures in E is an affine space modelled after the $\mathbb{C}$-vector space $A^{0,1}(\text{End } E)$.

Let $\mathscr{G}$ denote the group of C^∞ automorphisms of the bundle E. Given $D \in \mathscr{A}$ and $g \in \mathscr{G}$, we define a new holomorphic structure $D \cdot g \in \mathscr{A}$ by the prescription

$$D \cdot g = g^{-1} \circ D \circ g.$$

Any holomorphic structure D in E induces a holomorphic structure, also denoted D, in End E as follows

$$D: A^0(\text{End } E) \to A^{0,1}(\text{End } E)$$

$$f \mapsto [D, f] = D \circ f - f \circ D.$$

Having made this definition, we can write $D \cdot g = g^{-1} \circ (Dg) + D$.
Anyway, we now have a map

$$\mathscr{A} \times \mathscr{G} \to \mathscr{A}$$

$$(D, g) \mapsto D \cdot g.$$

This map is clearly an affine linear right action of $\mathscr{G}$ on $\mathscr{A}$.

The multiplicative subgroup $\mathbb{C}^*$ is embedded in $\mathscr{G}$ as the normal subgroup consisting of constant multiples of the identity automorphism of E. This subgroup $\mathbb{C}^*$ acts trivially on $\mathscr{A}$. Hence the above action induces an action of the group

$\mathscr{P}\mathscr{G} = \mathscr{G}/\mathbb{C}^*$ on $\mathscr{A}$. We now state a well-known integrability theorem, whose proof can be found in Donaldson [6].

PROPOSITION 3.2

Given a holomorphic structure $D \in \mathscr{A}$, there exists a unique structure of a holomorphic vector bundle E_D in E such that the $\bar{\partial}$-operator of E_D equals D.

Now we have the trivial

PROPOSITION 3.3

Let D_1 and D_2 be holomorphic structures in E. Then E_{D_1} and E_{D_2} are isomorphic as holomorphic vector bundles if and only if D_1 and D_2 lie in the same $\mathscr{G}$-orbit of $\mathscr{A}$.

Proof. The bundles E_{D_1} and E_{D_2} are isomorphic as holomorphic vector bundles if and only if there is a C^∞ isomorphism $g: E_{D_1} \rightarrow E_{D_2}$ which commutes with the $\bar{\partial}$-operators of the bundles, i.e., such that $g \circ D_1 = D_2 \circ g$. This means precisely that $g \in \mathscr{G}$ and $D_1 = D_2 \cdot g$.

DEFINITION 3.4

A holomorphic structure D in E is called a holomorphic π-structure if the map $D: A^0(E) \rightarrow A^{0,1}(E)$ is π-equivariant.

The set $\mathscr{A}^\pi$ of all holomorphic π-structures in E is an affine subspace of $\mathscr{A}$ in bijection with the subspace $A^{0,1}(\text{End } E)^\pi$ of $A^{0,1}(\text{End } E)$. Let $\mathscr{G}^\pi$ denote the subgroup of $\mathscr{G}$ consisting of π-automorphisms of E. It is easily seen that $\mathscr{A}^\pi$ is invariant under the action of $\mathscr{G}^\pi$ on $\mathscr{A}$. Further $\mathbb{C}^*$ is contained in $\mathscr{G}^\pi$, hence we get an action of $\mathscr{P}\mathscr{G}^\pi = \mathscr{G}^\pi_{/\mathbb{C}^*}$ on $\mathscr{A}^\pi$.

Remark 3.5. If V and W are C^∞ π-bundles on Y of the same rank, degree and local type (cf. Remark 2.10), then V and W are isomorphic as C^∞ π-bundles. Thus the set $\mathscr{A}^\pi/\mathscr{P}\mathscr{G}^\pi$ can be thought of as the set of isomorphism classes of holomorphic π-bundles on Y of rank r, degree d and of the local type as E.

DEFINITION 3.6

We say that a holomorphic π-structure D is stable if the corresponding holomorphic π-bundles E_D on Y is stable.

Let $\mathscr{A}^\pi_s$ denote the subset of $\mathscr{A}^\pi$ consisting of stable holomorphic π-structures. Then it is obvious that $\mathscr{A}^\pi_s$ is a $\mathscr{G}^\pi$-invariant subset. We know from Seshadri [20] that the isotropy subgroup of $\mathscr{G}^\pi$ at any $D \in \mathscr{A}^\pi_s$ is exactly $\mathbb{C}^*$. Thus the group $\mathscr{P}\mathscr{G}^\pi$ acts freely on $\mathscr{A}^\pi_s$. Let $M^\pi_s(Y) = \mathscr{A}^\pi_{s/\mathscr{P}\mathscr{G}^\pi}$. We state without proof.

PROPOSITION 3.7

The canonical projection $\varphi: \mathscr{A}^\pi_s \rightarrow M^\pi_s(Y)$ is a holomorphic principal $\mathscr{P}\mathscr{G}^\pi$-bundle.

Remark 3.8. We have not defined any topology or complex structure on $\mathscr{A}^\pi_s$ and $M^\pi_s(Y)$, so Proposition 3.7, as it stands, is meaningless. But using Sobolev spaces it is possible to give a natural complex manifold structure to $M^\pi_s(Y)$ and to make sense of Proposition 3.7. Since there are many accounts now of this kind of argument using

Sobolev spaces, we omit all the details and make a blanket reference to Narasimhan and Ramadas [23], Donaldson [6], Lübke and Okonek [12], Kim [10] and, Atiyah and Bott [1].

PROPOSITION 3.9

If $\varphi:\mathscr{A}^\pi_s \to M^\pi_s(Y)$ *denotes the quotient projection and if* $D\in\mathscr{A}^\pi_s$, *then the tangent space to* $M^\pi_s(Y)$ *at the point* $\varphi(D)$ *is naturally isomorphic to* $H^1\,(Y,\pi,\, \mathrm{End}\, E_D)$.

Remark 3.10. If V is a π-sheaf on Y, define $H^0(Y,\pi,V)$ to be the set of all π-invariant sections of V on Y. The correspondence

$$V\mapsto H^0(Y,\pi,V)$$

is then a left exact additive covariant functor from the abelian category of π-sheaves on Y to the category of abelian groups. For $i\geqslant 1$, the cohomology group $H^i(Y,\pi,V)$ is, by definition, the ith right derived functor of the above functor. It is known (cf Seshadri [20]) that $H^i(Y,\pi,V)$ is naturally isomorphic to $H^i(X,p^\pi_*\mathcal{O}(V))$, where $X=\pi\backslash Y$ and $p:Y\to X$ is the canonical projection.

Proof of Proposition 3.9. Fix $D\in\mathscr{A}^\pi_s$ and define

$$\sigma:\mathscr{P}\mathscr{G}^\pi\to\mathscr{A}^\pi_s.$$
$$g\mapsto D.g.$$

The Lie algebra of $\mathscr{P}\mathscr{G}^\pi$ is $\mathfrak{P}\mathfrak{G}^\pi = A^0(\mathrm{End}\ E)^\pi$. The differential of σ at the identity element of $\mathscr{P}\mathscr{G}^\pi$ is given by

$$D^\pi: A^0(\mathrm{End}\ E)^\pi\to A^{0,1}(\mathrm{End}\ E)^\pi$$
$$f\mapsto Df.$$

Thus the tangent space along the fibre of φ at D is precisely $\mathrm{Im}(D^\pi)$. The Hodge theory implies that

$$A^{0,1}(\mathrm{End}\ E)^\pi = \mathrm{Im}(D^\pi)\oplus\ker(D^{*,\pi}),$$

where $D^{*,\pi}$ is the Hermitian adjoint of D^π. Hence the tangent space to $M^\pi_s(Y)$ at $\varphi(D)$ is given by

$$\frac{\text{Tan. space to } \mathscr{A}^\pi_s \text{ at } D}{\text{Vertical tan. space}} = \frac{A^{0,1}(\mathrm{End}\ E)^\pi}{\mathrm{Im}(D^\pi)} \simeq \ker(D^{*,\pi}).$$

As usual, let $X=\pi\backslash Y$ and $p:Y\to X$ the canonical projection. Let $\alpha^{p,q}(\mathrm{End}\ E)$ denote the sheaf of $C^\infty(p,q)$-forms on Y with values in End E. Consider the sequence

$$0\to p^\pi_*\mathcal{O}(\mathrm{End}\ E_D)\to p^\pi_*\alpha^0(\mathrm{End}\ E)\xrightarrow{D^\pi} p^\pi_*\alpha^{0,1}(\mathrm{End}\ E)\to 0.$$

By averaging over π, this sequence is easily seen to be a fine resolution of $p^\pi_*\mathcal{O}(\mathrm{End}\ E_D)$

on X. Hence the sheaf theory implies that

$$\begin{aligned} H^1(X, p_*^\pi \mathcal{O}(\operatorname{End} E_D)) &= \operatorname{coker}(D^\pi : A^0(\operatorname{End} E)^\pi \to A^{0,1}(\operatorname{End} E)^\pi) \\ &= \ker D^{*,\pi} \\ &= T_{\varphi(D)}(M_s^\pi(Y)). \end{aligned}$$

On the other hand, we know (cf. Remark 3.10) that

$$H^1(X, p_*^\pi \mathcal{O}(\operatorname{End} E_D)) \simeq H^1(Y, \pi, \operatorname{End} E_D).$$

Thus we conclude that

$$T_{\varphi(D)}(M_s^\pi(Y)) = H^1(Y, \pi, \operatorname{End} E_D). \qquad \blacksquare$$

We recall now the definition of the determinant bundle on $\mathscr{A}$ as in Quillen [18]. For each $D \in \mathscr{A}$, define a complex line $\mathscr{L}_D$ by

$$\mathscr{L}_D = \lambda(\ker D)^* \otimes \lambda(\operatorname{coker} D),$$

where λ denotes the top exterior power. Since each D is an elliptic operator, the kernel and cokernel of D are finite dimensional, so $\mathscr{L}_D$ is well-defined. We thus have a family

$$\mathscr{L} = \bigcup_{D \in \mathscr{A}} \mathscr{L}_D \tag{3.1}$$

of lines parametrized by $\mathscr{A}$. It is a fact, proved by Quillen [18], that $\mathscr{L}$ is actually a holomorphic line bundle over $\mathscr{A}$. There is a right $\mathscr{G}$-action on $\mathscr{L}$ given by

$$\begin{aligned} &(D, (s_1^* \wedge \ldots \wedge s_k^*) \otimes (t_1 \wedge \ldots \wedge t_1)) \times g \\ &\quad \mapsto (D \cdot g, (g^{-1}(s_1^*) \wedge \ldots \wedge g^{-1}(s_k^*)) \otimes (g(t_1) \wedge \ldots \wedge g(t_1)) \end{aligned}$$

where $(s_1^*, \ldots, s_k^*)$ is a basis of $(\ker D)^*$ and $(t_1, \ldots, t_l)$ is a basis of coker D. If $g = \lambda \cdot id_E$ where $\lambda \in \mathbb{C}^*$, then $D \cdot g = D$ and g acts on $\mathscr{L}_D$ as

$$\begin{aligned} \xi &= (s_1^* \wedge \ldots \wedge s_k^*) \otimes (t_1 \wedge \ldots \wedge t_l) \\ &\quad \mapsto ((\lambda^{-1} s_1^*) \wedge \ldots \wedge (\lambda^{-1} s_k^*)) \otimes ((\lambda t_1) \wedge \ldots \wedge (\lambda t_l)) \\ &\quad = \lambda^{l-k} \cdot \xi = \lambda^{-\chi(E)} \cdot \xi \end{aligned}$$

where $\chi(E) = \operatorname{index}(D)$ which by Riemann–Roch, equals $d + r(1-g)$, d being the degree of E, r the rank of E and g the genus of Y.

We would like to modify the line bundle $\mathscr{L}$ on $\mathscr{A}$ in order to obtain a line bundle $\tilde{\mathscr{L}}$ on which the action of $\mathbb{C}^*$ is trivial. To do this, fix a point $y \in Y$ such that the isotropy subgroup π_y of y in π is trivial. Define

$$\tilde{\mathscr{L}} = \mathscr{L}^r \otimes \lambda(\mathscr{A} \times E_y)^{\chi(E)}, \tag{3.2}$$

where $\mathscr{L}^r$ denotes the rth tensor power of $\mathscr{L}$ and $\mathscr{A} \times E_y$ is the trivial vector bundle on $\mathscr{A}$ with fibre E_λ. The computation of the previous paragraph shows that $\mathbb{C}^*$ acts trivially on $\tilde{\mathscr{L}}$ and, hence, $\tilde{\mathscr{L}}$ becomes a $\mathscr{P}\mathscr{G}$-line bundle on $\mathscr{A}$. We denote the restriction of $\tilde{\mathscr{L}}$ to $\mathscr{A}_s^\pi$ by the symbol $\tilde{\mathscr{L}}$ itself. Since the natural projection

$$\varphi : \mathscr{A}_s^\pi \to M_s^\pi(Y)$$

is a holomorphic principal $\mathscr{P}\mathscr{G}^\pi$-bundle, the $\mathscr{P}\mathscr{G}^\pi$-line bundle $\tilde{\mathscr{L}}$ on $\mathscr{A}_s^\pi$ descends to a holomorphic line bundle L^π over $M_s^\pi(Y)$.

DEFINITION 3.11

The line bundle L^π is called the determinant bundle on the moduli space $M_s^\pi(Y)$.

Remark 3.12. If $\chi(E) = 0$, then the above argument shows that $\mathbb{C}^*$ acts trivially on the initially unmodified line bundle $\mathscr{L}$ (cf. equation (3.1)). Thus $\mathscr{L}$ itself descends to $M_s^\pi(Y)$ when the index of E is zero.

In the earlier part of this section we have constructed certain holomorphic objects. We would now like to study their metric properties. To do this we need the space of unitary connections and the unitary gauge group. Recall that we have fixed a π-invariant Hermitian metric h in the C^∞ vector bundle E.

DEFINITION 3.13

A unitary connection in E is a connection

$$\nabla : A^0(E) \to A^1(E)$$

such that for all s, $t \in A^0(E)$, we have

$$d(h(s,t)) = h(\nabla s, t) + h(s, \nabla t).$$

Let $\mathrm{End}(E,h)$ denote the C^∞ $\mathbb{R}$-vector bundle of skew-Hermitian endomorphisms of (E,h). It is easily checked that the set $\mathscr{C}$ of all unitary connections in E is an affine space modelled after the $\mathbb{R}$-vector space $A^1(\mathrm{End}(E,h))$. If ∇ is a unitary connection in E, the $(0,1)$-part D_∇ of ∇ is a holomorphic structure in E.

Let $\mathscr{U}$ denote the subgroup of $\mathscr{G}$ consisting of unitary automorphisms of (E,h). Given $\nabla \in \mathscr{C}$ and $g \in \mathscr{U}$, we define $\nabla \cdot g \in \mathscr{C}$ by

$$\nabla \cdot g = g^{-1} \circ \nabla \circ g.$$

This gives a right action of $\mathscr{U}$ on $\mathscr{C}$. There is a natural embedding $\lambda \mapsto \lambda \cdot id_E$ of $U(1)$ in $\mathscr{U}$. This subgroup $U(1)$ of $\mathscr{U}$ acts trivially on $\mathscr{C}$ and hence we get an action of $\mathscr{P}\mathscr{U} = \mathscr{U}_{/U(1)}$ on $\mathscr{C}$. The proof of the following simple proposition can be found in Kobayashi [11].

PROPOSITION 3.14

The $(0,1)$*-part map*

$$\mathscr{C} \to \mathscr{A}$$

$$\nabla \mapsto D_\nabla = (0,1)\text{-part of } \nabla$$

is a $\mathscr{P}\mathscr{U}$*-equivariant affine* $\mathbb{R}$*-linear isomorphism.*

DEFINITION 3.15

A unitary connection ∇ in E is called a unitary π-connection if the map $\nabla: A^0(E) \to A^1(E)$ is π-equivariant.

The set $\mathscr{C}^\pi$ of all unitary π-connections in E is an affine subspace of $\mathscr{C}$ isomorphic to the real subspace $A^1(\mathrm{End}\,(E,h))^\pi$ of $A^1(\mathrm{End}\,(E,h))$. Let $\mathscr{U}^\pi$ denote the subgroup of $\mathscr{U}$ consisting of unitary π-automorphisms of (E,h). It is easily seen that $\mathscr{C}^\pi$ is invariant under the action of $\mathscr{U}^\pi$ on $\mathscr{C}$. Further $U(1)$ is contained in $\mathscr{U}^\pi$, hence we get an action of $\mathscr{P}\mathscr{U}^\pi = \mathscr{U}^\pi/_{U(1)}$ on $\mathscr{C}^\pi$.

DEFINITION 3.16

A unitary π-connection ∇ in E is said to be reducible if there exists proper C^∞ π-subbundles E_1 and E_2 of E together with unitary π-connections ∇_1 and ∇_2 such that $E = E_1 \oplus E_2$ and $\nabla = \nabla_1 \oplus \nabla_2$. If ∇ is not reducible we say that ∇ is irreducible.

DEFINITION 3.17

A unitary π-connection ∇ is said to be Einstein–Hermitian if the curvature $R(\nabla)$ of ∇ satisfies the identity

$$R(\nabla) = \frac{2\pi \cdot \mu(E)}{i \cdot |\pi|} \cdot \omega \otimes id_E$$

where ω is the Kähler form on (Y, μ), $|\pi|$ is the order of the group π and

$$\mu(E) = \frac{\mathrm{degree}(E)}{\mathrm{rank}(E)}.$$

DEFINITION 3.18

A holomorphic π-structure $D \in \mathscr{A}^\pi$ is said to be Einstein–Hermitian if the corresponding connection is so. A holomorphic π-structure is called indecomposable if there are no proper holomorphic π-subbundles E_1 and E_2 of E_D such that $E_D = E_1 \oplus E_2$.

The following theorem is a special case of an important theorem of Simpson [21]. This also follows from Seshadri [20].

Theorem 3.19. *An indecomposable holomorphic π-structure $D \in \mathscr{A}^\pi$ is stable if and only if there exists an Einstein–Hermitian holomorphic π-structure in the $\mathscr{P}\mathscr{G}^\pi$-orbit of D. Further if D_1 and D_2 are two Einstein–Hermitian holomorphic π-structures in the $\mathscr{P}\mathscr{G}^\pi$-orbit of D, then D_1 and D_2 lie in the same $\mathscr{P}\mathscr{U}^\pi$-orbit.*

Let $\mathscr{C}_s^\pi$ denote the subset of $\mathscr{C}^\pi$ consisting of irreducible Einstein–Hermitian π-connections. Then it is obvious that $\mathscr{C}_s^\pi$ *is a* $\mathscr{U}^\pi$-invariant subset. It can be easily checked (cf. Kobayashi [11]) that the stabilizer in $\mathscr{U}^\pi$ at any $\nabla \in \mathscr{C}_s^\pi$ is exactly $U(1)$. Thus the group $\mathscr{P}\mathscr{U}^\pi$ acts freely on $\mathscr{C}_s^\pi$. Let $N_s^\pi = \mathscr{C}_s^\pi / \mathscr{P}\mathscr{U}^\pi$. Then we have

PROPOSITION 3.20

The canonical projection $\psi: \mathscr{C}_s^\pi \to N_s^\pi$ is a C^∞ principal $\mathscr{P}\mathscr{U}^\pi$-bundle.

Remark 3.21. The comments made in Remark 3.8 apply to the above Proposition too. We will assume Proposition 3.20 as proved and quote the same references as in Remark 3.8.

Let ∇ be an Einstein–Hermitian holomorphic π-connection in E. Then the curvature $R(\tilde{\nabla})$: $A^0(\mathrm{End}(E,h)) \to A^2(\mathrm{End}(E,h))$ of the induced connection $\tilde{\nabla}$ in $\mathrm{End}(E,h)$ is given by

$$\begin{aligned} R(\tilde{\nabla})(f) &= [R(\nabla), f] = [\alpha \otimes id_E, f], \quad \alpha \in A^{1,1} \\ &= \alpha \otimes [id_E, f] = 0. \end{aligned}$$

Thus $\tilde{\nabla}$ is a flat π-connection in the real vector bundle $\mathrm{End}(E,h)$. We denote $\tilde{\nabla}$ also by ∇. The vector bundle $\mathrm{End}(E,h)$ together with the flat connection $\tilde{\nabla} = \nabla$ becomes a local system of real vector spaces on Y which we shall denote by $\mathrm{End}\,\mathscr{E}_\nabla$. In Seshadri [19], the de Rham and Hodge theories are developed for C^∞ forms with values in local systems. We denote the de Rham cohomology groups of Y with values in $\mathrm{End}\,\mathscr{E}_\nabla$ by $H^i_d(Y, \mathrm{End}\,\mathscr{E}_\nabla)$. These cohomology groups are finite dimensional vector spaces. The π-invariants in $H^i_d(Y, \mathrm{End}\,\mathscr{E}_\nabla)$ are denoted by $H^i_d(Y, \pi, \mathrm{End}\,\mathscr{E}_\nabla)$.

We now compute the tangent space of the space N^π_s.

Lemma 3.22: *The tangent space to $\mathscr{C}^\pi_s$ at a point $\nabla \in \mathscr{C}^\pi_s$ equals the group of cocycles*

$$Z^i_d(Y, \pi, \mathrm{End}\,\mathscr{E}_\nabla) = \mathrm{Ker}\{\nabla^\pi : A^1(\mathrm{End}(E,h))^\pi \to A^2(\mathrm{End}(E,h))^\pi\}$$

Proof. Let $\mathscr{C}^\pi_0$ denote the set of irreducible π-connections. $\mathscr{C}^\pi_0$ is an open set in $\mathscr{C}^\pi$. Let V be the $\mathbb{R}$-subspace of $A^2(\mathrm{End}(E,h))$ given by

$$V = \{\alpha \otimes id_E / i\alpha \in A^{1,1}_{\mathbb{R}}\},$$

where $A^{1,1}_{\mathbb{R}}$ denotes the space of real (1, 1)-forms on Y. Let

$$B : A^2(\mathrm{End}(E,h))^\pi \to \frac{A^2(\mathrm{End}(E,h))^\pi}{V}$$

denote the natural projection and define

$$\begin{aligned} &\theta : \mathscr{C}^\pi_0 \to \frac{A^2(\mathrm{End}(E,h))^\pi}{V} \\ &\quad \nabla_0 \mapsto B(R(\nabla_0)). \end{aligned}$$

We now have the Hodge decomposition

$$A^2(\mathrm{End}(E,h))^\pi = \mathrm{Im}(\nabla^\pi_0) \oplus \ker(\nabla^{*,\pi}_0).$$

We claim that $\ker(\nabla^{*,\pi}_0) = V$. To see this, suppose $\nabla^{*,\pi}_0(\alpha) = 0$ for some $\alpha \in A^2(\mathrm{End}(E,h))^\pi$. Then $*^{-1} \circ \nabla_0 \circ *(\alpha) = 0$. But the irreducibility of ∇_0 implies that the parallel sections of ∇_0 in $\mathrm{End}(E,h)$ are exactly the scalars. Therefore $*(\alpha) = \lambda \cdot id_E$ for some $\lambda \in \mathbb{C}$. Hence $*^2(\alpha) = \lambda \cdot \omega \otimes id_E$ i.e., $\alpha = -\lambda \cdot \omega \otimes id_E$. Since α is skew-Hermitian, $i\lambda \in \mathbb{R}$. This proves that $\ker(\nabla^{*,\pi}_0) = V$. Therefore, the Hodge decomposition above implies that

$$\frac{A^2(\mathrm{End}\,(E,h))}{V} \simeq \mathrm{Im}(\nabla_0^\pi).$$

We conclude from this that θ is a submersion on $\mathscr{C}_0^\pi$. Clearly $\mathscr{C}_s^\pi = \theta^{-1}(0)$. Thus the tangent space to $\mathscr{C}_0^\pi$ at any point ∇ is given by the kernel of $d\theta$ at ∇, i.e., equals $\ker(\nabla^\pi)$. This is precisely $Z_d^1(Y,\pi,\mathrm{End}\,\mathscr{E}_\nabla)$. ■

Recall that $\psi:\mathscr{C}_s^\pi \to N_s^\pi$ is the canonical quotient projection.

PROPOSITION 3.23

The tangent space to N_s^π at a point $\psi(\nabla)$, $\nabla\in\mathscr{C}_s^\pi$, is given by $H_d^1(Y,\pi,End\,\mathscr{E}_\nabla)$.

Proof. As in the proof of Proposition 3.9, we can check that the vertical tangent space at $\nabla\in\mathscr{C}_s^\pi$ is

$$B_d^1(Y,\pi,\mathrm{End}\,\mathscr{E}_\nabla) = \{\nabla\alpha/\alpha\in A^0(\mathrm{End}(E,h))^\pi\}.$$

Thus the tangent space to N_s^π at $\psi(\nabla)$ is

$$\begin{aligned}\frac{\text{Tan. space to } \mathscr{C}_s^\pi \text{ at } \nabla}{\text{Vertical tan. sp. at } \nabla} &= \frac{Z^1(Y,\pi,\mathrm{End}\,\mathscr{E}_\nabla)}{B_d^1(Y,\pi,\mathrm{End}\,\mathscr{E}_\nabla)}\\ &= H_d^1(Y,\pi,\mathrm{End}\,\mathscr{E}_\nabla).\end{aligned}$$ ■

We shall identify the real tangent space of $M_s^\pi(Y)$ with its holomorphic tangent space. Theorem 3.19 implies that if $\nabla\in\mathscr{C}_s^\pi$, then the (0, 1)-part D_∇ of ∇ belongs to $\mathscr{A}_s^\pi$ and the map $\mathscr{C}_s^\pi\to\mathscr{A}_s^\pi$, $\nabla\mapsto D_\nabla$ induces a bijection $I:N_s^\pi\to M_s^\pi(Y)$ between the quotients. Clearly I is C^∞. The differential of I at $\psi(\nabla)$, $\nabla\in\mathscr{C}_s^\pi$, is given by

$$\begin{aligned}dI:H_d^1(Y,\pi,\mathrm{End}\,E_\nabla)&\to H^1(Y,\pi,\mathrm{End}\,E_{D_\nabla}) = \ker(D^{*,\pi}),\\ a&\mapsto\alpha''\end{aligned}$$

where α is the harmonic representative of the class a and α'' is the (0, 1)-part of α. Clearly dI is an isomorphism. Thus $I:N_s^\pi\to M_s^\pi(Y)$ is a local diffeomorphism, hence a diffeomorphism.

We now define a Hermitian metric on $M_s^\pi(Y)$. If $\varphi:\mathscr{A}_s^\pi\to M_s^\pi(Y)$ denotes the canonical projection and $D\in\mathscr{A}_s^\pi$, define a Hermitian inner product

$$\begin{array}{rcccl}\langle .,.\rangle_{\varphi(D)}: & T_{\varphi(D)}(M_s^\pi(Y)) & \times & T_{\varphi(D)}(M_s^\pi(Y)) & \to \mathbb{C}\\ & \| & & \| & \\ & \ker(D^{*,\pi}) & \times & \ker(D^{*,\pi}) & \to \mathbb{C}\end{array}$$

by

$$\langle\alpha,\beta\rangle = \frac{1}{i}\int_Y \mathrm{tr}(\alpha\wedge\beta^*).$$

As $\varphi(D)$ varies $\langle .,.\rangle_{\varphi(D)}$ gives a smooth Hermitian metric on $M_s^\pi(Y)$. We shall denote its fundamental (1, 1)-form by Θ^π.

Define a real 2-form $\tilde{\Omega}$ on $\mathscr{C}_s^\pi$ by

$$\tilde{\Omega}_\nabla(\alpha, \beta) = \int_Y tr(\alpha \wedge \beta),$$

where $\nabla \in \mathscr{C}^\pi$ and $\alpha, \beta \in A^1(\mathrm{End}(E, h))^\pi$. It is easily seen that $\tilde{\Omega}$ is a $\mathscr{P}\mathscr{U}^\pi$ invariant symplectic form on $\mathscr{C}^\pi$. Restricted to the submanifold $\mathscr{C}_s^\pi$, the form $\tilde{\Omega}$ vanishes along the vertical direction of ψ. This can be seen as follows: any vertical vector α at $\nabla \in \mathscr{C}_s^\pi$ is of the form $\alpha = \nabla f$ where $f \in A^0 \mathrm{End}(E, h)^\pi$; so if $\beta \in Z_d^1(Y, \pi, \mathrm{End}\, \mathscr{E}_\nabla)$, we have

$$\begin{aligned}
\tilde{\Omega}(\alpha, \beta) &= \int_Y \mathrm{tr}(\nabla f \wedge \beta)\\
&= \int_Y \mathrm{tr} \nabla(f \nabla \beta) - \int_Y \mathrm{tr}(f \nabla \beta)\\
&= \int_Y d\, \mathrm{tr}(f\beta) - 0 \quad (\text{since } \nabla\beta = 0)\\
&= 0 \qquad (\text{by Stokes theorem})
\end{aligned}$$

Thus $\tilde{\Omega}$ descends to a symplectic form Ω on N_s^π.

PROPOSITION 3.24

The Hermitian metric on $M_s^\pi(Y)$ is Kähler.

Proof. It is easily seen that $I^*(\Theta) = \Omega$, hence Θ is closed, as Ω is closed and I is a diffeomorphism.

Consider the determinant bundle $\mathscr{L}$ and $\mathscr{A}$ (cf. equation 3.1). For each $D \in \mathscr{A}$, let

$$0 \leqslant \lambda_{D,1} \leqslant \lambda_{D,2} \leqslant \ldots.$$

be the eigenvalues of the Laplacian

$$\Box_D := D^* \circ D : A^\circ(E) \to A^0(E),$$

and define the zeta function

$$\zeta_D(s) = \sum_{\lambda_{D,j} > 0} (\lambda_{D,j})^{-s}.$$

This series converges for $\mathrm{Re}\, s > 1$ and the function $\zeta_D(s)$ extends to a meromorphic function on $\mathbb{C}$ which is holomorphic at 0. Define a Hermitian inner product

$$h_Q : \mathscr{L}_D \times \mathscr{L}_D \to \mathbb{C},$$

$$\text{by } h_Q = \exp(-\zeta_D'(0)).\langle .,. \rangle_{L^2}$$

where $\langle .,. \rangle_{L^2}$ is the L^2-inner product in $\mathscr{L}_D$ induced by the metric in E. It is proved in Quillen [18] that h_Q is a smooth Hermitian metric in $\mathscr{L}$.

There is also a natural Hermitian inner product on the affine space $\mathscr{A} \simeq A^{0,1}(\mathrm{End}\, E)$

which is given by

$$\langle \alpha, \beta \rangle = \frac{1}{i} \cdot \int_Y \mathrm{tr}(\alpha \wedge \beta^*)$$

for $\alpha, \beta \in A^{0,1}(\mathrm{End}\, E)$. This gives rise to a Kähler metric in $\mathscr{A}$ whose Kähler form is denoted here by $\tilde{\Theta}$. We state the following theorem of Quillen [18].

Theorem 3.25. *The Chern form of the canonical connection in the holomorphic Hermitian line bundle* $(\mathscr{L}, h_Q)$ *on* $\mathscr{A}$ *equals the Kähler form* $\tilde{\Theta}$.

Consider the two principal bundles

$$\begin{tikzcd} \mathscr{C}^\pi_s \arrow[r, "\tilde{I}"] \arrow[d, "\psi"'] & \mathscr{A}^\pi_s \arrow[d, "\varphi"] \\ N^\pi_s \arrow[r, "I"'] & M^\pi_s(Y). \end{tikzcd} \tag{3.3}$$

On the line bundle $\tilde{\mathscr{L}}$ (cf. equation 3.2) we have a metric given by the tensor product of the Quillen metric h_Q in $\mathscr{L}$ and the given metric in E_y. Since $\lambda(E_y)$ is a flat unitary bundle on $\mathscr{A}$, the Chern form of the metrized line bundle $\tilde{\mathscr{L}}$ is r times the Chern form of $\mathscr{L}$ and, hence, by Theorem 3.25, equals $r.\tilde{\Theta}$. Let $\tilde{\mathscr{M}}$ be the pull-back of $\tilde{\mathscr{L}}$ to $\mathscr{C}^\pi_s$ by the map $\tilde{I}$. Then $\tilde{\mathscr{M}}$ is a C^∞ $\mathscr{P}\mathscr{U}^\pi$-line bundle on $\mathscr{C}^\pi_s$ and the pull-back metric and the pull-back connection in the line bundle $\tilde{\mathscr{M}}$ are invariant under the action of $\mathscr{P}\mathscr{U}^\pi$ on $\mathscr{C}^\pi_s$. Thus $\tilde{\mathscr{M}}$ descends to a C^∞ Hermitian line bundle M on N^π_s with a Hermitian connection. This bundle M is naturally isomorphic to $I^*(L^\pi)$ (cf. definition 3.11) as a Hermitian line bundle. It follows from this that L^π naturally gets a metric and a connection. This connection is easily seen to be compatible with the metric and the holomorphic structure in L^π. We denote the metric in L^π by g.

DEFINITION 3.26

We call g the Quillen metric in L^π.

Theorem 3.27. *The Chern form of the Quillen metric in the holomorphic line bundle* L^π *equals* r *times the Kähler form* Θ^π *on* $M^\pi_s(Y)$.

Proof. Consider the commutative diagram (3.3). We have

$$\begin{aligned} (I \circ \psi)^* c_1(L^\pi, g) &= c_1((I \circ \psi)^* L^\pi, (I \circ \psi)^* g) \\ &= c_1(\tilde{\mathscr{M}}, \tilde{I}^* \nabla), \quad \text{where } \nabla = \text{canonical connection in } \tilde{\mathscr{L}} \\ &= \tilde{I}^* c_1(\tilde{\mathscr{L}}) \\ &= \tilde{I}^*(r \cdot \tilde{\Theta}) \quad \text{(cf. the remarks following the commutative diagram 3.3)} \\ &= r \cdot \tilde{\Omega} \\ &= (I \circ \psi)^*(r \cdot \Theta). \end{aligned}$$

Since $I\circ\psi$ is a submersion, this implies that

$$c_1(L,g) = r\cdot\Theta. \qquad \blacksquare$$

4. The relation between the π-determinant and the parabolic determinant

In this section we show that the isomorphism from the moduli space of π-bundles to the moduli space of parabolic bundles carries the π-determinant L^π (cf. Definition 3.11) to a multiple of an "intrinsic" line bundle on the parabolic moduli. By "intrinsic" we mean that the line bundle depends only on the parabolic data and not on the choice of the π-structure. We refer to this intrinsic line bundle as the parabolic determinant bundle. Here we consider only the bundle structures and leave the comparison of the metric structures to the next section. The main results we use here are the Grothendieck–Riemann–Roch theorem and a theorem of Deligne *et al* ([5] and [3]) on the determinant of the tensor product of two families of line bundles.

Consider a compact Riemann surface Y and let π be a finite group acting effectively and holomorphically on Y. Let $X = {}_{\pi\backslash}Y$ be the quotient Riemann surface and $p\colon Y\to X$ the canonical projection. The map p is a ramified covering. Let I be the set of ramification points of p in Y and let $J = p(I)$ be the set of critical values of p in X. Both I and J are finite sets. For each $x\in J$, choose a point $y_x\in p^{-1}(x)$ and let n_x be the order of π_x, the isotropy subgroup of π at y_x. Fix once and for all, for each $x\in J$, a set $C_x\subseteq\pi$ of left coset representatives of π_x.

DEFINITION 4.1

A holomorphic family of compact Riemann surfaces is a proper surjective holomorphic submersion $f\colon\mathscr{X}\to T$ of complex manifolds such that for each $t\in T$, the submanifold $X_t = f^{-1}(t)$ is a Reimann surface.

Let $f\colon\mathscr{X}\to T$ be a family of Riemann surfaces and let $\mathscr{F}$ be a coherent $\mathcal{O}_{\mathscr{X}}$-module. Then, by the Proper Mapping Theorem, all the higher direct images $R^if_*\mathscr{F}$ are coherent $\mathcal{O}_T$-modules so,

$$f_!(\mathscr{F}) := \sum(-1)^i R^if_*\mathscr{F}$$

is a well defined element of $K(T)$. Therefore

$$d(\mathscr{F}) = \lambda(f_!\mathscr{F})$$

is a holomorphic line bundle over T, called the determinant bundle of $\mathscr{F}$. The fibre of $d(\mathscr{F})$ at $t\in T$ is canonically isomorphic to

$$\lambda(H^0(X_t,\mathscr{F}_t))^*\otimes\lambda(H^1(X_t,\mathscr{F}_t)),$$

where $\mathscr{F}_t = \mathscr{F}|_{X_t}$.

Remark 4.2. Suppose $f\colon\mathscr{X}\to T$ is a family of Riemann surfaces. Then, the determinant construction defined above enjoys several nice properties. For instance, if

$$0\to\mathscr{F}'\to\mathscr{F}\to\mathscr{F}''\to 0$$

is an exact sequence of coherent $\mathcal{O}_{\mathscr{X}}$-modules, then

$$d(\mathscr{F}) = d(\mathscr{F}') \otimes d(\mathscr{F}'') = d(\mathscr{F} \oplus \mathscr{F}'').$$

Hence, when we consider the determinants $d(\mathscr{F})$ we will assume that all short exact sequences of $\mathcal{O}_{\mathscr{X}}$-modules split.

DEFINITION 4.3

A family of vector bundles on the Riemann surface Y parametrized by a complex manifold T is, by definition, a holomorphic vector bundle over $Y \times T$. A family of π-bundles on Y parametrized by T is a holomorphic π-bundle on $Y \times T$, where the action of π on T is trivial.

DEFINITION 4.4

A family of parabolic bundles on X parametrized by a manifold T is a holomorphic vector bundle F on $X \times T$ together with the following data: for each $x \in J$, we are given a filtration of the vector bundle $F^x = F|_{x \times T}$ by subbundles

$$F^x = F^1 F^x \supseteq F^2 F^x \supseteq \ldots \supseteq F^{k_x} F^x \supseteq F^{k_x+1} F^x = 0.$$

We denote $\mathrm{Gr}^j F^x = F^j F^x / F^{j+1} F^x$.

Suppose E is a family of π-bundles on Y parametrized by a manifold T. Let r be the rank of E. For each $t \in T$, the local type of the π-bundle $E_t = E|_{Y \times t}$ on Y at each point y_x is independent of t and is given by integers

$$0 \leqslant l_{x,1} < \ldots < l_{x,k_x} \leqslant n_x - 1 \text{ and } (r_{x,1}, \ldots, r_{x,k_x})$$

such that

$$\sum_{j=1}^{k_x} r_{x,j} = r.$$

Define $\alpha_{x,j} = l_{x,j}/n_x$ and $n_{x,j} = n \cdot \alpha_{x,j}$ where $n = |\pi|$. Let $F \to X \times T$ be the family of parabolic bundles induced by E (cf. Proposition 2.11), namely, $F = E^\pi$ is the holomorphic vector bundle on $X \times T$ corresponding to the locally free $\mathcal{O}_{X \times T}$-module $(p \times 1_T)_*^\pi \mathcal{O}(E)$ together with an induced parabolic structure. We now compute the determinant of E in terms of F and the parabolic structure of F:

PROPOSITION 4.5

Consider a family E of π-bundles on Y parametrized by a complex manifold T and let $F = E^\pi$ be the induced parabolic family on $X \times T$, as above. Then as line bundles on T, we have

$$d(F) = d((p \times 1_T)^* F) \otimes \left(\bigotimes_{\substack{x \in J \\ j = 1, \ldots, k_x}} \lambda(\mathrm{Gr}^j F^x)^{-n_{x,j}} \right)$$

Proof. We first note that for each $x \in J$ and $j = 1, \ldots, k_x$, $\mathrm{Gr}^j F^x \cong E_j^x$ as vector bundles on T where $E^x = E|_{y_x \times T}$ and $F^x = F|_{x \times T}$ and $E^x = \sum_{j=1}^{k_x} E_j^x$ is the eigenspace decomposition of E^x, i.e, on each E^x_j, π_x acts as a character of weight $l_{x,j}$. We prove this by producing a natural homomorphism

$$h_i\colon \mathrm{Gr}^j F^x \to E^x_j.$$

The map h_j is defined as follows: Given $0 \neq v \in \mathrm{Gr}^j F^x$, let $\bar{v} \in F^x$ represent v. Now take any section s of F over some open set containing x which satisfies the condition that $s(x) = \bar{v}$. Let $\bar{s}$ be the invariant section E over an open set containing y_x, which represents s. If $\bar{s}$ is the image of $t \otimes \mathbf{1}_i$ where t is a section of E such that $t(y_x) \neq 0$ and $\mathbf{1}_i$ is the canonical section of $\mathcal{O}(-iy_x)$, under the morphism of sheaves

$$0 \to E \otimes \mathcal{O}(-iy_x) \to E,$$

define

$$h_j = t(y_x).$$

From the correspondence between π-bundles and parabolic bundles as described by Mehta and Seshadri [13] it follows that h_j is an isomorphism.

Therefore to prove the proposition, we need to verify that

$$d(E) = d((p \times 1)^* F)\Bigg(\bigotimes_{\substack{x \in J \\ j = 1,\dots,k_x}} \lambda(E^x_j)^{-n_{x,j}} \Bigg). \tag{4.1}$$

We have a canonical $\mathcal{O}_{Y \times T}$-module monomorphism

$$0 \to (p \times 1)\mathcal{O}(F) \to \mathcal{O}(E).$$

The cokernel $\mathcal{T}$ of this monomorphism is a torsion $\mathcal{O}_{Y \times T}$-module supported on $I \times T$ where I denotes the ramification locus in Y. We thus have an exact sequence of $\mathcal{O}_{y \times T}$-modules

$$0 \to (p \times 1)^* \mathcal{O}(F) \to \mathcal{O}(E) \to \mathcal{T} \to 0. \tag{4.2}$$

By Remark 4.2, we get

$$\begin{aligned} d(E) &= d((p \times 1)^* F) \cdot d(\mathcal{T}) \\ &= d((p \times 1)^* F) \cdot \Bigg(\bigotimes_{y \in I} \lambda(\mathcal{T}^y)^{-1} \Bigg) \end{aligned} \tag{4.3}$$

where $\mathcal{T}^y = \mathcal{T}|_{y \times T}$. But $\mathcal{T}$ is a π-sheaf on $Y \times T$, hence for each $y \in Y$ and $\gamma \in \pi$, $\mathcal{T}^y$ is canonically isomorphic to $\mathcal{T}^{\gamma \cdot y}$ on T. If $x \in J$ and $y \in p^{-1}(x)$, there is a unique $\gamma \in C_x$ such that $y = \gamma \cdot y_x$ (recall that we have fixed a section $C_x \subseteq \pi$ of $\pi \to \pi/\pi_x$); hence we have a canonical isomorphism $\mathcal{T}^y \cong \mathcal{T}^{y_x}$. Thus for each $x \in J$,

$$\bigotimes_{y \in p^{-1}(x)} \lambda(\mathcal{T}^y) = \lambda(\mathcal{T}^{y_x})^{n/n_x}$$

so that

$$\bigotimes_{y \in J} \lambda(\mathcal{T}^y) = \bigotimes_{x \in J} \lambda(\mathcal{T}^{y_x})^{n/n_x}.$$

Using this (4.3) becomes

$$d(E) = d((p \times 1)^* F)\cdot\left(\bigotimes_{x\in J} \lambda(\mathscr{T}^{y_x})^{-n/n_x}\right). \tag{4.4}$$

From (4.1) and (4.4) we see that it suffices to show that for each $x \in J$,

$$\lambda(\mathscr{T}^{y_x})^{n/n_x} = \bigotimes_{j=1}^{k_x} \lambda(E_j^x)^{n_{x,j}}. \tag{4.5}$$

Since $n_{x,j} = n\cdot l_{x,j}/n_x$, this amounts to verifying that

$$\lambda(\mathscr{T}^{y_x}) = \bigotimes_{j=1}^{k_x} \lambda(E_j^x)^{l_{x,j}}. \tag{4.6}$$

Proving (4.6) for a fixed $x \in J$ is a local problem at y_x. By Remark (2.10), using a π-compatible coordinate chart at y_x we are reduced to the following situation: Let π be the group of nth roots of unity acting on the unit disc D by multiplication and let $p\colon \to D = {}_{\pi\backslash}D$ be the quotient map $p(z) = z^n$. Let V be a vector bundle on the variety T and define an action of π on V by $\zeta(v) = \zeta^l\cdot v$ for each $\zeta \in \pi$ and $v \in V$, where $0 \leqslant l \leqslant n-1$ is a fixed integer. Consider the π-bundle $W = p_T^* V$ on $D \times T$ where $p_T\colon \to T$ is the projection map and let $\mathscr{T}$ denote the cokernel of $0 \to (p\times 1)^*(p\times 1)_*^{\pi}\mathcal{O}(W) \to \mathcal{O}(W)$. Now, prove that

$$\lambda(\mathscr{T}^{\mathrm{o}}) = \lambda(V)^{-l}, \tag{4.7}$$

where o is the centre of D. Let us check that (4.7) is true. We can easily see that $(p\times 1)^*(p\times 1)_*^{\pi}\mathcal{O}(W) = \mathfrak{m}^l\cdot\mathcal{O}(W)$ where $\mathfrak{m}$ is the ideal sheaf of $\mathrm{o}\times T$ in $D \times T$. Therefore the sheaf $\mathscr{T}$ is isomorphic to the cokernel of

$$0 \to \mathfrak{m}^l\cdot\mathcal{O}(W) \to \mathcal{O}(W).$$

Now we have an exact sequence

$$0 \to \mathfrak{m} \to \mathcal{O}_{D\times T} \to \mathscr{S} \to 0,$$

where $\mathscr{S} = \mathcal{O}_{D\times T}/\mathfrak{m}$. Tensoring this short exact sequence with $\mathfrak{m}^{j-1}\mathcal{O}(W), j = 1,\ldots,l$, we get a system of exact sequences

$$0 \to \mathfrak{m}^j\mathcal{O}(W) \to \mathfrak{m}^{j-1}\mathcal{O}(W) \to \mathfrak{m}^{j-1}\mathcal{O}(W)\otimes\mathscr{S} \to 0, \quad j = 1,\ldots,l.$$

By general functionality, it follows that $\lambda(\mathscr{T}^{\mathrm{o}}) = \bigotimes_{j=1}^{l} \lambda(\mathscr{T}_j^{\mathrm{o}})$ where $\mathscr{T}_j = \mathfrak{m}^{j-1}\mathcal{O}(W)\otimes\mathscr{S}$ and $\mathscr{T}_j^{\mathrm{o}} = \mathscr{T}|_{\mathrm{o}\times T}$. But $\mathscr{T}_j^{\mathrm{o}} = (L^{\mathrm{o}})^{j-1}\otimes V$ where L is the line bundle on $D \times T$ associated to the sheaf $\mathfrak{m}$ and $L^{\mathrm{o}} = L|_{\mathrm{o}\times T}$. Since L^{o} is trivial on T, we get $\mathscr{T}_j^{\mathrm{o}} = V$ for $j = 1,\ldots,l$. Thus $\lambda(\mathscr{T}^{\mathrm{o}}) = \lambda(V)^l$ which is precisely (4.7). This proves the proposition.

We now construct parabolic determinants. For a more detailed description of these determinant bundles we refer the reader to Narasimhan and Ramadas [15]. Let X be a compact Riemann surface and $J \subseteq X$ a finite set. Fix integers k_x and

$(r_{x,1},\ldots,r_{x,k_x})$ such that $\Sigma_{j=1}^{k_x} r_{x,j} = r$. Let $0 \leqslant \alpha_{x,1} \leqslant \ldots \leqslant \alpha_{x,k_x} \leqslant 1$ be rational numbers for each $x \in J$. Let $M_j^p(X)$ denote the moduli space of stable parabolic bundles of rank r and degree d with weights $\alpha_{x,j}$, and multiplicities $r_{x,j}$.

Suppose $F \to X \times T$ is a family of parabolic bundles of type $\{\alpha_{x,j}, r_{x,j}\}_{x \in J}$, i.e., $rk(\mathrm{Gr}^j F^x) = r_{x,j}$ and the weights are $\alpha_{x,j}$. Then a possible candidate $pd(F)$ for the parabolic determinant is the usual determinant minus a factor involving the flags and the weights; thus, we define $\overline{pd}(F)$ to be the element of $\mathrm{Pic}(T)_{\mathbb{Q}} = \mathrm{Pic}(T) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ given by

$$\overline{pd}(F) = d(F) - \sum_{\substack{x \in J \\ j=1,\ldots,k_x}} \alpha_{x,j} \cdot \lambda(\mathrm{Gr}^j F^x).$$

Now if L is a line bundle on T and $G = F \otimes p_T^*(L)$ where $p_T: X \times T$ is the projection, then F and G are, by definition, equivalent families. Therefore we would like to have $\overline{pd}(F) = \overline{pd}(G)$ in $\mathrm{Pic}(T)_{\mathbb{Q}}$. But $d(G) = d(f) - \chi(E) \cdot L$ and $\lambda(\mathrm{Gr}^j G^x) = \lambda(\mathrm{Gr}^j F^x) + r_{x,j} \cdot L$ in $\mathrm{Pic}(T)_{\mathbb{Q}}$ where $\chi(E) = \chi(E_t)$ is independent of t. Hence $\overline{pd}(G) = \overline{pd}(F) - \chi_p(F) \cdot L$, where

$$\chi_p(F) = \chi(F) + \sum_{\substack{x \in J \\ j=1,\ldots,k_x}} r_{x,j} \cdot \alpha_{x.j}.$$

We thus see that the definition of $\overline{pd}(F)$ has to be modified to get an invariant determinant. Fix once and for all a point $x_0 \in X \backslash J$.

DEFINITION 4.7

Let $F \to X \times T$ be a family of parabolic bundles of type $\{r_{x,j}, \alpha_{x,j}\}$ on X parametrized by a variety T. Then the parabolic determinant $\overline{pd}(F)$, of F is the element of $\mathrm{Pic}(T)_{\mathbb{Q}} = \mathrm{Pic}(T) \otimes \mathbb{Q}$ defined by the formula

$$\overline{pd}(F) = \mathrm{d}(F) - \sum_{\substack{x \in J \\ j=1,\ldots,k_x}} \alpha_{x,j} \lambda(\mathrm{Gr}^j F^x) + \frac{\chi_p(F)}{r} \lambda(F^{x_0})$$

where

$$\chi_p(F) = \chi(F_t) + \sum_{x,j} \alpha_{x,j} r_{x,j}.$$

The above discussion shows that if F and G are equivalent families of parabolic bundles on $X \times T$, then $\overline{pd}(F) = \overline{pd}(G)$ in $\mathrm{Pic}(T)_{\mathbb{Q}}$.

We would like to think of the parabolic determinant as a concrete line bundle representing an element of $\mathrm{Pic}(T)$, rather than as an abstract entity in $\mathrm{Pic}(T)_{\mathbb{Q}}$. To achieve this we have to multiply the $\overline{pd}(F)$ of Definition 4.7 by an integer to remove the denominators from the rationals $\alpha_{x,j}$ and $\chi_p(F)/r$. Write $\alpha_{x,j} = p_{x,j}/q_{x,j}$ where $p_{x,j}$ and $q_{x,j}$ are relatively prime positive integers if $\alpha_{x,j} > 0$ and $q_{x,j} = 1$ if $\alpha_{x,j} = 0$. Denote by m the l.c.m of $\{q_{x,j}/x \in J, 1 \leqslant j \leqslant k_x\}$ and let $m_{x,j} = m \cdot \alpha_{x,j}$. Then $0 \leqslant m_{x,1} < m_{x,2} < \ldots < m_{x,k_x} \leqslant m - 1$ are integers for each $x \in J$.

DEFINITION 4.8

Let $F \to X \times T$ be a family of parabolic bundles of type $\{\alpha_{x,j}, r_{x,j}\}$. Then the parabolic determinant bundle, $pd(F)$, of F is the line bundle on T defined by

$$pd(F) = \mathrm{d}(E)^{rm} \cdot \left\{ \bigotimes_{\substack{x \in J \\ j = 1, \ldots, k_x}} \lambda(\mathrm{Gr}^j F^x)^{-m_{x,j} \cdot r} \right\} \cdot \lambda(F^{x_0})^{m \cdot \chi_p(F)}.$$

Note that in $\mathrm{Pic}(T)_{\mathbb{Q}}$, $pd(F) = rm \cdot \overline{pd}(F)$.

Remark 4.9. The above arguments again imply that $pd(F) = pd(G)$ if F and G are equivalent families of parabolic bundles on $X \times T$.

We recall a proof from Narasimhan and Ramadas [15, § 2.c] that there is a parabolic determinant bundle on the moduli space $M_s^p(X)$. By tensoring with a fixed line bundle we may assume that the degree d is large enough to ensure that for every $E \in M_s^p(X)$, $H^0(E)$ generates E and $H^1(E) = 0$. Let Q denote the Quot scheme of coherent $\mathcal{O}_X$-modules which are quotients of $\mathcal{O}_X^k$, $k = d + r(1 - g)$, with Hilbert polynomial $P(m) = d + r(m - g + 1)$. Then there is a Poincaré family of sheaves on $X \times Q$. Denote by R the open subset of Q consisting of locally free sheaves E such that $H^0(\mathcal{O}_X^k) \to H^0(E)$ is an isomorphism. Let $\mathscr{F}$ denote the restriction of the Poincaré family to $X \times R$. For each $x \in J$, let Flag $\mathscr{F}^x$ denote the flag bundle of type $(r_{x,1}, \ldots, r_{x,1})$ associated to the vector bundle $\mathscr{F}^x$ on R; Flag $\mathscr{F}^x$ is a fibre bundle over R. Let $\tilde{R}$ denote the fibre product $\tilde{R} = \prod_{\substack{R \\ x \in J}} \text{Flag}\, \mathscr{F}^x$. Then $\tilde{R}$ parametrizes a family $\tilde{\mathscr{F}}$ of parabolic bundles on X. Let $\tilde{R}_s$ denote the subset of $\tilde{R}$ corresponding to stable parabolic bundles. There is a natural action of $\mathrm{PGL}(k)$ on $\tilde{R}_s$ and the canonical map $\tilde{R}_s \to M_s^p(X)$ is a holomorphic principal $\mathrm{PGL}(k)$-bundle. Let $pd(\tilde{\mathscr{F}})$ denote the determinant bundle (cf. Definition 4.8.) on R_s defined by the family $\tilde{\mathscr{F}} \to X \times \tilde{R}_s$. Clearly the action of $GL(k)$ on $\tilde{R}_s$ lifts to an action on $pd(\tilde{\mathscr{F}})$. Let us see how the scalars $\mathbb{C}^* \subseteq GL(k)$ act on $pd(\mathscr{F})$. From Definition 4.8 and the properties of R, we see that

$$pd(\mathscr{F}) = \lambda(\mathcal{O}_{\tilde{R}_s})^{-rm} \cdot \left\{ \bigotimes_{\substack{x \in J \\ j = 1, \ldots, k_x}} \lambda(\mathrm{Gr}^j \mathscr{F}^x)^{-rm_{x,j}} \right\} \cdot \lambda(\mathscr{F}^{x_0})^{m \cdot \chi_p(\mathscr{F})}.$$

Thus a scalar $\lambda \in \mathbb{C}^*$ acts on $pd(\tilde{\mathscr{F}})$ as multiplication by λ^c, where

$$c = -rmk - r \sum_{\substack{x \in J \\ j = 1, \ldots, k_x}} m_{x,j} \cdot r_{x,j} + rm\chi_p(\tilde{\mathscr{F}});$$

on simplifying the right side, we see that $c = 0$. Therefore $pd(\tilde{\mathscr{F}})$ is a $PGL(k)$-line bundle on $\tilde{R}_s$ and hence descends to a holomorphic line bundle L^p on $M_s^p(X)$.

DEFINITION 4.10

The above line bundle L^p on $M_s^p(X)$ is called the parabolic determinant bundle on $M_s^p(X)$.

Consider a family of parabolic bundles on X, $F \to X \times T$, parametrized by a variety

T. Choose a compact Riemann surface Y and a group π of order n acting on Y such that (i) the quotient ${}_{\pi} Y$ is biholomorphic to X; and (ii) if $p: Y \to X$ denotes the quotient projection, then for each $x \in J$ and $y \in p^{-1}(x)$, the order of the isotropy subgroup of π at y equals m. Then n is a multiple of m and we let $a = n/m$. Define

$$n_{x,j} = n \cdot \alpha_{x,j} = a \cdot m_{x,j} \quad \text{for } x \in J \text{ and } j = 1, \dots, k_x.$$

By Proposition 2.11, the family F induces a family of π-bundles on Y, $E \to Y \times T$, parametrized by T. Let $\pi d(E)$ denote the determinant bundle of E as in §3. Fix a point $y \in p(x)$. Since $x \in X \backslash J$, y is not a ramification point of p and

$$\pi d(E) = d(E) \otimes \lambda(E^{y_0})^{\chi(E)}$$

where $\chi(E) = \chi(E_t)$ is independent of t. The two line bundles $pd(F)$ and $\pi d(E)$ on T are related by

PROPOSITION 4.11

Let F and E be as above. Then as elements of $Pic(T)_{\mathbb{Q}} = Pic(T) \otimes \mathbb{Q}$,

$$\{\pi d(E) \cdot pd(F)^{-a}\}^2 = \bigotimes_{x \in J} \{\lambda(F^x)\lambda(F^{x_0})^{-1}\}^{ra(m-1)}.$$

Before proving the above proposition we make a simple observation whose proof is due to V. Balaji:

Lemma 4.12. *Let $f: \mathscr{X} \to T$ be a family of compact Riemann surfaces and suppose that F and G are holomorphic vector bundles on $\mathscr{X}$ with ranks r and n, respectively. Then*

$$d(F \otimes G) d(F)^{-n} d(G)^{-r} = d(\lambda(F) \cdot \lambda(G)) d(\lambda(F))^{-1} d(\lambda(G))^{-1}$$

in $Pic(T) \otimes \mathbb{Q}$.

Proof. Denote by $A^*(T) = \sum_{i=0}^{\infty} A^i(T)$ the Chow ring of T and for any $\alpha \in A^*(T)$, let $\{\alpha\}_i$ denote the degree i component of α in $A^i(T)$. Also let $f_*: A^*(\mathscr{X}) \to A^*(T)$ denote integration along the fibre. Then the Grothendieck–Riemann–Roch (G-R-R) theorem implies that for any vector bundle V on $\mathscr{X}$,

$$c_1(d(V)) = f_*(\{ch(V) \cdot td(T_f)\}_2),$$

where T_f denotes the tangent bundle of $\mathscr{X}$ along the fibres of f. Note that T_f is a line bundle on $\mathscr{X}$. Let $\alpha_1, \dots, \alpha_r$ be the Chern roots of F and $\beta_1, \dots, \beta_n$ the Chern roots of G. Then $(\alpha_i + \beta_j)$ are the Chern roots of $F \otimes G$. Using the splitting principle (cf. Fulton [8], page 54) we formally compute

$$\{ch(F) \cdot td(T_f)\}_2 = \frac{r\mu^2}{12} + \left(\sum_{i=1}^{r} \alpha_i\right)\frac{\mu}{2} + \frac{1}{2}\sum_{i=1}^{r} \alpha_i^2$$

where $\mu = c_1(T_f)$. Similarly

$$\{ch(F) \cdot td(T_f)\}_2 = \frac{r\mu^2}{12} + \left(\sum_{j=1}^{r} \beta_j\right)\frac{\mu}{2} + \frac{1}{2}\sum_{j=1}^{r} \beta_i^2.$$

Finally,

$$\begin{aligned}\{ch(F\otimes G)\cdot td(T_f)\}_2 &= \frac{nr\mu^2}{12} + \left(\sum_{i=1}^{r}\sum_{j=1}^{n}(\alpha_i+\beta_j)\right)\cdot\frac{\mu}{2} \\ &\quad + \frac{1}{2}\left(\sum_{i=1}^{r}\sum_{j=1}^{n}(\alpha_i+\beta_j)\right)^2 \\ &= \frac{nr\mu^2}{12} + n(\Sigma\alpha_i)\frac{\mu}{2} + r(\Sigma\beta_j)\frac{\mu}{2} + r(\Sigma\beta_j)\frac{\mu}{2} + \frac{n}{2}\sum_i \alpha_i^2 \\ &\quad + \frac{r}{2}\sum_j \beta_j^2 + (\Sigma\alpha_i)(\Sigma\beta_j)\end{aligned}$$

i.e.,

$$\begin{aligned}\{ch(F\otimes G)\cdot td(T_f)\} &= \{n\cdot ch(F)\cdot td(T_f) + rch(G)\cdot td(T_f)\}_2 \\ &\quad + c_1(F)\cdot c_1(G) - \frac{nr\mu^2}{12}.\end{aligned}$$

Applying G-R-R and noting that $f_*(\mu^2)=0$, we get

$$c_1(d(F\otimes G)) = n\cdot c_1(d(F)) + rc_1(d(G)) + f_*(c_1(F)\cdot c_1(G)),$$

i.e.,

$$c_1(d(F\otimes G))\cdot d(F)^{-n}d(G)^{-r}) = f_*(c_1(F)\cdot c_1(G)).$$

Replacing F by $\lambda(F)$ and G by $\lambda(G)$ in the above relation, we get

$$c_1(d(\lambda(F)\cdot\lambda(G))\cdot d(\lambda(F))^{-1}d(\lambda(G))^{-1}) = f_*(c_1(F)\cdot c_1(G))$$

since $c_1(\lambda(F)) = c_1(F)$. Thus

$$c_1(d(F\otimes G)\cdot d(F)^{-n}\cdot d(G)^{-r}) = c_1(d(\lambda(F)\lambda(G))d(\lambda(F)^{-1}d(\lambda(G)^{-1}).$$

Since $c_1:\mathrm{Pic}(T)\otimes\mathbb{Q}\to A^*(T)\otimes\mathbb{Q}$ is an isomorphism (Fulton [8], page 294) our lemma is proved.

We now recall a formalism due to Deligne [5]; we quote below Proposition 2.5 and Lemma 2.6 from Beilinson and Manin [3] as a brief summary of this formalism.

PROPOSITION 4.13

Let $f:\mathscr{X}\to T$ be a holomorphic family of compact Riemann surfaces. Then for each pair L, M of line bundles on $\mathscr{X}$, there is a line bundle $\langle L,M\rangle$ on T with the following properties:

(i) *There are canonical isomorphisms*

$$\langle L,\mathcal{O}_{\mathscr{X}}\rangle = \mathcal{O}_T,\quad \langle L_1\otimes L_2,M\rangle = \langle L_1,M\rangle\langle L_2,M\rangle,$$
$$\langle L,M\rangle = \langle M,L\rangle,\quad \langle L,M^{-1}\rangle = \langle L,M\rangle^{-1}$$

(ii) *If $M=\mathcal{O}_x(D)$, where D is a relatively positive divisor on $\mathscr{X}$, locally free on T, then*

$$\langle L,M\rangle = d(L\otimes\mathcal{O}_D)^{-1}\cdot d(\mathcal{O}_D).$$

(iii) *There is a canonical isomorphism*

$$\langle L,M\rangle^{-1} = d(L\otimes M)\cdot d(\mathcal{O}_{\mathscr{X}})\cdot d(L)^{-1}d(M)^{-1}.$$

Now we proceed to the Proof of Proposition 4.11. Using Proposition 4.5 we see that

$$d(E) = d((p \times 1_T)^* F) \cdot \bigotimes_{x \in J} \lambda(\mathrm{Gr}^j F^x)^{-n_{x,j}}. \tag{4.3}$$

But $H^i(X, p_* V) = H^i(Y, V)$ canonically, so

$$d((p \times 1_T)^* F) = d((p \times 1)_* (p \times 1)^* F) \tag{4.4}$$

Now, by the projection formula

$$(p \times 1)_* (p \times 1)^* F = F \otimes G,$$

where

$$G = (p \times 1)_* \mathcal{O}_{Y \times T} = p_X^* p_* \mathcal{O}_Y \otimes p_T^* \mathcal{O}_T,$$

and $p_X : X \times T \to X$ and $p_T : X \times T \to T$ are the projections. Thus (4.4) becomes

$$d((p \times 1)^* F) = d(F \otimes G). \tag{4.5}$$

We now prove from Lemma 4.12 that

$$d(F \otimes G) \cdot d(F)^{-n} \cdot d(G)^{-r} = \langle \lambda(F), \lambda(G) \rangle^{-1} \cdot d(\lambda(F))^{-1} d(\lambda(G))^{-1}$$

in $\mathrm{Pic}(T)_{\mathbb{Q}}$. Using Proposition 4.13 (iii), we write this as

$$d(F \otimes G) \cdot d(F)^{-n} \cdot d(G)^{-r} = \langle \lambda(F), \lambda(G) \rangle^{-1} \cdot d(\mathcal{O}_{X \times T})^{-1}.$$

But $d(G)$ and $d(\mathcal{O}_{X \times T})$ are trivial on T. Thus

$$\langle \lambda(F), \lambda(G) \rangle^{-1} = d(F \otimes G) \cdot d(F)^{-n} \tag{4.6}$$

in $\mathrm{Pic}(T)_{\mathbb{Q}}$. Squaring both sides of (4.6) and using Proposition 4.13, we get

$$\langle \lambda(F), \lambda(G)^2 \rangle = \{d(F \otimes G) \cdot d(F)^{-n}\}^2. \tag{4.7}$$

But $\lambda(G) = \lambda(p_X^* p_* \mathcal{O}_Y) \cdot p_T^* (\mathcal{O}_T)^n$. Therefore,

$$\begin{aligned} \langle \lambda(F), \lambda(G)^2 \rangle &= \langle \lambda(F), \lambda(p_X^* p_* \mathcal{O}_Y)^2 \cdot (p_T^* \mathcal{O}_T)^{2n} \rangle \\ &= \langle \lambda(F), \lambda(p_X^* p_* \mathcal{O}_Y)^2 \rangle \langle \lambda(F), p_T^* \mathcal{O}_T \rangle^{2n} \\ &= \langle \lambda(F), p_X^* \lambda(p_* \mathcal{O}_Y P)^2 \rangle \end{aligned}$$

because $\langle \lambda(F), p_T^* \mathcal{O}_T \rangle$ is trivial by virtue of Proposition 4.13. Now $\lambda(p_* \mathcal{O}_Y)^2 = \mathcal{O}_X(D)$ where $D = -\sum_{x \in J} a(m-1) \cdot x$ (cf. Beauville *et al* [2], §2.5). Therefore, by Proposition 4.13(ii)

$$\langle \lambda(F), \lambda(G)^2 \rangle = \langle \lambda(F), p_X^* \mathcal{O}_X(D) \rangle = d(\lambda(F) \otimes p_X^* \mathcal{O}_D)^{-1} d(p_X^* \mathcal{O}_D)$$

i.e.,

$$\langle \lambda(F), \lambda(G)^2 \rangle = \bigotimes_{x \in J} \lambda(F^x)^{a(m-1)} \tag{4.8}$$

where we drop $d(p_X^*\mathcal{O}_D)$ as it is trivial on T. Combining (4.7) and (4.8), and using (4.5),

$$\{d((p\times 1)^*F)\cdot d(F)^{-n}\}^2 = \bigotimes_{x\in J}\lambda(F^x)^{a(m-1)}.$$

Now apply (4.3) we obtain

$$\left\{d(E)\cdot\bigotimes_{x\in J}\lambda(\mathrm{Gr}^j F^x)^{n_{x,j}}\cdot d(F)^{-n}\right\}^2 = \bigotimes_{x\in J}\lambda(F^x)^{a(m-1)}.$$

Raising the above equation to the rth power and noting that $n_{x,j}=a\cdot m_{x,j}$ and $n=a\cdot m$, we see that

$$\{\pi d(E)\cdot\lambda(E^{y_0})^{-\chi(E)})\cdot(pd(F)^{-a}\cdot\lambda(F^{x_0})^{\chi_p(F)\cdot n})\}^2 = \bigotimes_{x\in J}\lambda(F^x)^{ra(m-1)}. \tag{4.9}$$

Since y_o is not a ramification point of p and $p(y_o)=x_o$, we have $F^{y_0}\cong E^{x_0}$. So we can write (4.9) as

$$\{\pi d(E)\cdot pd(F)^{-a}\}^2\cdot\lambda(F^{x_0})^{2n\chi_p(F)-2\chi(E)} = \bigotimes_{x\in J}\lambda(F^x)^{ra(m-1)}. \tag{4.10}$$

The Riemann–Hurwitz formula gives

$$2n\chi_p(F)-2\chi(E)=|J|ra(m-1).$$

So we may express (4.10) as

$$\{\pi d(E)\cdot pd(F)^{-a}\}^2=\bigotimes_{x\in J}(\lambda(F^x)\lambda(F^{x_0})^{-1})^{ra(m-1)}$$

n $\mathrm{Pic}(T)_{\mathbb{Q}}$. This completes the proof of Proposition 4.11.

Let us see what Proposition 4.11 implies for our moduli spaces. Fix a line bundle δ of degree d on X and let $M_s^p(X,\delta)$ be the smooth subvariety of $M_s^p(X)$ consisting of parabolic bundles F such that $\lambda(F)\cong\delta$. Similarly, let $M_s^\pi(Y,\delta)$ be the smooth submanifold of $M_s^\pi(Y)$ consisting of π-bundles E such that $\lambda(E^\pi)\cong\delta$. Then Proposition 2.11 gives a natural isomorphism $\theta: M_s^p(X,\delta)\to M_s^\pi(Y,\delta)$. In §3, we constructed a π-determinant L^π on $M_s^\pi(Y,\delta)$ using Quillen's theorem. In this section we defined the parabolic determinant bundle L^p on $M_s^p(X,\delta)$. These determinants are related to each other by

PROPOSITION 4.14

The two line bundles $\theta^*(L^\pi)$ *and* $(L^p)^a$, *where* a *is the integer defined in the paragraph following Definition* 4.10, *are isomorphic on* $M_s^p(X,\delta)$.

Proof. Consider the variety $\tilde{R}_s$ mentioned earlier on; $\tilde{R}_s$ sits as a principal $PGL(k)$-bundle over $M_s^p(X)$ and parametrizes a family $\mathscr{F}$ of parabolic bundles on X. Let $\tilde{V}$ be the corresponding family of π-bundles on Y parametrized by $\tilde{R}_s$. For each pair of points $x, y \in X$, let $\tilde{G}_{x,y}$ be the line bundle on $\tilde{R}_s$ given by

$$\tilde{G}_{x,y} = \lambda(\tilde{\mathscr{F}}^x)\lambda(\tilde{\mathscr{F}}^x)^{-1}.$$

We easily see that $\tilde{G}_{x,y}$ is a $PGL(k)$-line bundle and descends to a line bundle $G_{x,y}$ on $M_s^p(X)$. By Proposition 4.11,

$$(\pi d(\tilde{V}) \cdot pd(\tilde{\mathscr{F}})^{-a})^2 = \bigotimes_{x \in J} (\tilde{G}_{x,x_o})^{ra(m-1)} \tag{4.11}$$

in $\mathrm{Pic}(\tilde{R}_s) \otimes \mathbb{Q}$. But (cf. Drezet and Narasimhan [7], Lemma 4.2) $\mathrm{Pic}_G(\tilde{R}_s) \to \mathrm{Pic}(\tilde{R}_s)$ is injective, where $\mathrm{Pic}_G(\tilde{R}_s)$ is the group of $PGL(k)$-isomorphism classes of $PGL(k)$-line bundles on $\tilde{R}_s$. Therefore, since all the bundles in (4.11) lie in $\mathrm{Pic}_G(\tilde{R}_s)$, we see that (4.11) holds in $\mathrm{Pic}_G(\tilde{R}_s) \otimes \mathbb{Q}$. By descending to $M_s^p(X)$, we get

$$(\theta^* L^\pi (L^p)^{-a})^2 = \bigotimes_{x \in J} (G_{x,x_0})^{ra(m-1)}. \tag{4.12}$$

Consider a fixed $x \in X$. As y varies in X, $G_{x,y}$ varies in a continuous family; since $\mathrm{Pic}_0 M_s^p(X, \delta)$ is trivial, we conclude that $G_{x,y}$ is trivial on $M_s^p(X, \delta) \forall x, y \in X$. Thus (3.12) implies that

$$(\theta^* L^\pi (L^p)^{-a})^2 = 1$$

in $\mathrm{Pic}(M_s^p(X, \delta)) \otimes \mathbb{Q}$. Since $\mathrm{Pic} M_s^p(X, \delta)$ is torsion-free, we conclude that

$$\theta^* L^\pi = (L^p)^a$$

on $M^p(X, \delta)$. ■

5. The metric on the parabolic determinant

In this section we define a Hermitian metric on the parabolic determinant bundle and give a proof of Theorem 1.2.

Let $M_s^p(X)$ and $M_s^p(X, \delta)$ be as in § 4. Again, choose a covering $p: Y \to X$ of degree n. We continue using the notation of § 4.

Suppose $E \in M_s^p$. Then $\mathrm{End}\, E$ is local system on $X_0 = X \backslash J$. We denote this local system by $\underline{\mathrm{End\,E}}$. Let $H_c^1(X_0, \underline{\mathrm{End\,E}})$ denote the compactly supported de Rham cohomology of X_0 with values in the local system $\underline{\mathrm{End\,E}}$. The real tangent space to $M_s^p(X)$ at E is canonically isomorphic to the image I_E of the natural map

$$H_c^1(X_0, \underline{\mathrm{End\,E}}) \to H_d^1(X_0, \underline{\mathrm{End\,E}}),$$

where H_d^1 denotes usual de Rham cohomology. We shall prove this fact in Lemma 5.5 below; for the moment let us assume it.

We define a sympletic form Ω_E^p on $T_E(M_s^p(X))$ by

$$\Omega_E^p(a, b) = \int_{X_0} \mathrm{tr}(\alpha \wedge \beta),$$

where $a, b \in I_E$ and α, β are compactly supported forms representing a and b. As E varies over $M_s^p(X)$, Ω_E^p vary to define a C^∞ symplectic form Ω^p on $M_s^p(X)$.

Recall that we have an isomorphism $\theta: M_s^p(X) \to M_s^\pi(Y)$. In §3 we defined a symplectic form, which we now denote as Ω^π, on $M_s^\pi(Y)$. The two forms Ω^π and Ω^p are related as follows.

Lemma 5.1 $\theta^*(\Omega^\pi) = n \cdot \Omega^p$.

Proof. Recall that the tangent space to $M_s^\pi(Y)$ at a point $E \in M_s^\pi(Y)$ is given by $H_d^1(Y, \pi, \underline{\operatorname{End} E})$ where $\underline{\operatorname{End} E}$ is the local system defined by the Einstein-Hermitian connection in E. Let $F = E^\pi$. Then the differential of θ at E is described as follows. Let $a \in T_E(M_s^\pi(Y))$. Choose a π-invariant 1-form α on y with values in $\underline{\operatorname{End} E}$ representing a. Then $\alpha|_{Y \setminus I}$ descends to a form on $X_0 = X \setminus J$ which is cohomologous to a compactly supported form $\alpha_\#$ on X_0 with values in $\underline{\operatorname{End} E}$. Thus $\alpha_\#$ defines an element $a_\#$ in $I_F = T_F(M_s^p(X))$. The differential of θ^{-1} at E is given by

$$d\theta^{-1}: T_E(M_s^\pi(Y)) \to T_F(M_s^p(X))$$
$$a \to a_\#.$$

Now, let $a, b \in T_E(M_s^\pi(Y))$. Then

$$\begin{aligned}(\theta^{-1})^*(\Omega^p)(a,b) &= \Omega^p(d\theta^{-1}(a), d\theta^{-1}(b)) \\ &= \Omega^p(a_\#, b_\#) \\ &= \int_{X_0} \operatorname{tr}(\alpha_\# \wedge \beta_\#) \\ &= \frac{1}{n}\int_Y \operatorname{tr}(\alpha \wedge \beta) \\ &= \frac{1}{n}\Omega^\pi(a,b).\end{aligned}$$

Thus

$$(\theta^{-1})^*\Omega^p = \frac{1}{n}\Omega^\pi,$$

i.e.,

$$\Omega^p = \frac{1}{n}\theta^*\Omega^\pi. \qquad \blacksquare$$

Let us now state the obvious.

Lemma 5.2. *Let V and W be two complex manifolds and suppose $f: V \to W$ is a biholomorphism. Let $\beta: T(W)_{\mathbb{C}} \times T(W)_{\mathbb{C}} \to \mathbb{C}$ be a real positive* (1, 1,)*-form on W and let $\beta_{\mathbb{R}}$ denote $\beta|_{T(W) \times T(W)}$. Suppose $\alpha: T(V) \times T(V) \to \mathbb{R}$ is a real 2-form on V such that $f^*\beta_{\mathbb{R}} = \alpha$. Then the complexification $\alpha_{\mathbb{C}}$ of α is a real, positive* (1, 1)*-form on V.*

Thus the complexification Θ^p of the 2-form Ω^p on $M_s^p(X)$ is real, positive and of type (1, 1), so it defines a Kähler structure on $M_s^p(X)$.

In §3, we defined a natural Hermitian metric h^π in the determinant bundle L^π on

$M_s^p(Y)$. This metric induces a metric $\theta^* h^\pi$ in the pull-back $\theta^* L^\pi$ over $M_s^p(X)$. But Proposition 4.14 implies that $\theta^* L^\pi = (L^p)^a$ on $M_s^p(X,\delta)$. Thus we have obtained a metric in $(L^p)^a$ on $M_s^p(X,\delta)$. This defines a metric h^p in L^p. We now have Theorem 1.1 of the Introduction.

Theorem 5.3. *The Chern form of the metrized line bundle (L^p, h^p) equals mr times the Kähler form Θ^p on $M_s^p(X,\delta)$.*

Proof. Since both $c_1(L^p,h^p)$ and $mr\cdot\Theta^p$ are real (1, 1)-form, it suffices to show that they are equal on the real tangent space of $M_s^p(X,\delta)$. But as real forms

$$\begin{aligned} a\cdot c_1(L^p,h^p)_{\mathbb{R}} &= c_1((L^p)^a,(h^p)^a) = c_1(\theta^* L^\pi, \theta^* h^\pi)_{\mathbb{R}} = \theta^* c_1(L^\pi, h^\pi)_{\mathbb{R}} \\ &= \theta^*(r\cdot\Omega^\pi) = rn\cdot\Omega^p = rn\cdot\Theta^p_{\mathbb{R}}, \end{aligned}$$

using Theorem 3.27. Dividing by a and noting that $m\cdot a = n$, we get

$$c_1(L^p,h^p)_{\mathbb{R}} = rm\cdot\Theta^p_{\mathbb{R}}.$$

This proves the Theorem. ■

The metric a priori depends on the choice of the covering (Y,π). But we now see that in fact it is independent of π up to a positive constant.

PROPOSITION 5.4

Let (Y_1,π_1) and (Y_2,π_2) be two ramified coverings of X with the properties of Y as above. Let h_1^p and h_2^p be the induced metrics in L^p. Then there is a constant $c > 0$ such that $h_1^p = c\cdot h_2^p$ in L^p over $M_s^p(X,\delta)$..

Proof. Since h_1^p and h_2^p are two Hermitian metrics in the same line bundle L^p, there exists a positive c^∞ function $f{:}M_s^p(X,\delta)\to\mathbb{R}^+$ such that $h_1 = f\cdot h_2$. By theorem 5.3, h_1 and h_2 have the same curvature. So, locally, $\partial\bar\partial\log h_1 = \partial\bar\partial\log h_2$, and hence, $\partial\bar\partial\log f = 0$. Thus $\log f$ is a plurisubharmonic function on $M_s^p(X,\delta)$. Now the moduli space $M^p(X,\delta)$ of semistable parabolic bundles is a normal projective variety and codim of the complement of $M_s^p(X,\delta)\geqslant 2$, so $\log f$ extends to a plurisubharmonic function on the whole of $M^p(X,\delta)$. Since $M^p(X,\delta)$ is compact, $\log f = a$ is constant, and $h_1 = ch_2$ where $c = e^a$. ■

In the rest of the section we prove.

Lemma 5.5.. *The real tangent space to $M_s^p(X)$ at E is canonically isomorphic to the image I_E of the natural map*

$$H_c^1(X_c, \underline{\mathrm{End\,E}}) \to H_d^1(X_0, \underline{\mathrm{End\,E}}).$$

Proof. It is enough to prove the lemma in the case where parabolic degree is zero. If the parabolic degree is not zero then choose sufficiently large number of points $a_1,\ldots,a_n\in X_0$ and include them among the parabolic points and give on the fibre of each a_i the trivial quasi-parabolic structure; i.e. the only flag is the whole fibre (cf.

Mehta and Seshadri [13], Remark 1.17). Assign non-zero weights to the flags so that the new parabolic degree is zero. This gives an isomorphism of $M_s^p(X)$ to some other moduli of parabolic stable bundles with parabolic degree 0. Let V denote the image of E under this isomorphism. Stable bundles of parabolic degree 0 arise as representations of the fundamental group of the complement of the parabolic points in $U(n)$. So, as V is a degree 0 stable bundle End V has a structure of a local system. But around each a_i the monodromy is trivial; scalars are in the kernel of the adjoint representation. So End E and End V are naturally isomorphic on X_0; in particular End E has a structure of a local system on X_0. Now onwards we will assume the parabolic degree to be zero.

So, let X be a compact Riemann surface of genus $g \geqslant 2$ and $J \subseteq X$ a finite set. Fix integers $d \in \mathbb{Z}$ and $r \geqslant 1$. Let us recall our earlier notation. For each $x \in J$, let $0 \leqslant \alpha_{x,1} < \alpha_{x,2} < \cdots < \alpha_{x,k_x} < 1$ be rational numbers and let $(r_{x,1}, \ldots, r_{x,k_x})$ be positive integers such that $\sum_{j=1}^{k_x} r_{x,j} = r$. Assume that

$$d + \sum_{\substack{x \in J \\ j=1,\ldots,r}} \alpha_{x,j} \cdot r_{x,j} = 0.$$

Let $M_s^p(X)^0$ denote the moduli space of stable parabolic bundles of rank r, degree d and weights $\alpha_{x,j}$ with multiplicities $r_{x,j}$. For each $x \in J$, define a function $c_x : \{1, \ldots, r\} \to [0, 1]$ by

$$c_x(i) = \alpha_{x,j}, \quad \text{if } r_{x,1} + \cdots + r_{x,j-1} + 1 \leqslant r_{x,1} + \cdots + r_{x,j}.$$

Let W_x denote the conjugacy class in $U(r)$ of the diagonal matrix

$$\begin{pmatrix} e^{2\pi i c_x(1)} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i c_x(r)} \end{pmatrix}.$$

Express the $\alpha_{x,j}$ with a common denominator m, $\alpha_{x,j} = m_{x,j}/m$. Then there exists a discrete subgroup Γ of $PSL_2(\mathbb{R})$ acting on the upper half-plane H with the following properties: (i) Γ has no parabolic fixed points; (ii) $\Gamma \backslash H \cong X$; (iii) Γ is generated by $2g + |J|$ elements $\{a_1, \ldots, a_g, b_1, \ldots, b_g, c_x, x \in J\}$ with the relations

$$\left(\prod_{i=1}^{g} [a_i, b_i]\right)\left(\prod_{x \in J} c_x\right) = 1 \text{ and } c_x^m = 1, \ x \in J.$$

Let R denote the manifold of irreducible unitary representations $\rho : \Gamma \to V(r)$ such that $\rho(c_x) \in W_x$ for all $x \in J$. Then by Mehta and Seshadri [13], $R \cong M_s^p(X)^0$.

The real tangent space to R at a representation ρ is equal to $H^1(\Gamma, \mathrm{ad}\rho)$ where ad: $U(r) \to \operatorname{Lie} U(r)$ is the adjoint representation. Using an argument of Prasad [17], we shall identify this group cohomology with compactly supported cohomology. From Prasad [17, Lemma 3 and §6] we get $H^1(\Gamma, \mathrm{ad}\rho) = H^1(X, A)$ where A is the invariant direct image of the constant sheaf Lie $U(r)$ on H. Let $X_0 = X \backslash J$ and let $i : X_0 \to X$ denote the inclusion map. It is easily seen that $i_*(A|_{X_0}) = A$. Therefore the edge homomorphism associated to the Leray spectral sequence of i implies that the canonical map $g: H^1(X, A) \to H^1(X_0, A)$ is injective. On the other hand Theorem 4.10.1 of Godement [9] applied to the closed set J gives an exact sequence

$$\cdots \to H_c^1(X_0, A) \to H^1(X, A) \to H^1(J, A) \to \cdots$$

where H_c^1 denotes cohomology with compact supports. But J being a finite set, we get $H^1(J, A) = 0$. Thus the map $f: H_c^1(X_0, A) \to H^1(X, A)$ is surjective. Let

$$h: H_c^1(X_0, A) \to H^1(X_0, A)$$

denote the canonical map. Then we have a commutative diagram

$$\begin{tikzcd}
H_c^1(X, A) \arrow[r, "f"] \arrow[d, "h"'] & H^1(X, A) \arrow[d, "g"] \\
H^1(X_0, A) \arrow[r, dash, "\text{identity}"'] & H^1(X_0, A)
\end{tikzcd}$$

Since f is surjective and g injective, we conclude that $H^1(X, A) \cong \mathrm{Im}(h)$, i.e., $T_\rho(R)$ is the image of $H_c^1(X_0, A)$ in $H^1(X_0, A)$ under the canonical map h. This completes the proof of the Lemma. ■

6. An application

In this section we deduce a finiteness statement from our earlier results.

Let Y be a compact Riemann surface of genus at least 2. Fix two relatively prime integers r and d with $r \geqslant 1$. Let $M(Y)$ denote the moduli space of stable bundles of rank r and degree d on Y. Suppose a finite group π acts holomorphically and effectively on Y. Let $M^\pi(Y)$ denote the moduli space of stable π-bundles of rank r, degree d and fixed local type on Y. Then both $M(Y)$ and $M^\pi(Y)$ are projective algebraic manifolds and there is a canonical forgetful map

$$\alpha: M^\pi(Y) \to M(Y).$$

PROPOSITION 6.1

The morphism α is a finite map.

Proof. As in § 3, we can construct a determinant line bundle L on M and show that L is positive (cf. Theorem 3.27). In fact, to see this we put π to be the trivial group everywhere in § 3. Since $M(Y)$ is compact, the Kodaira embedding theorem implies that L is ample. But the determinant bundle L^π on $M^\pi(Y)$ is exactly equal to $\alpha^*(L)$. Thus α pulls back the ample line bundle L on $M(Y)$ to an ample line bundle on $M^\pi(Y)$. As a result α is a finite morphism. ■

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Convolution properties of some classes of meromorphic univalent functions

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Abstract. Convolution technique and subordination theorem are used to study certain class of meromorphic univalent functions in the punctured unit disc.

Keywords. Meromorphic starlike; subordination; Hadamard product.

1. Introduction

In this article, we study certain subclasses of functions of the form

$$g(z)=\frac{1}{z}+\sum_{n=0}^{\infty} b_n z^n \tag{1.1}$$

which are univalent and analytic in $0<|z|<1$. Thus, at $z=0$, these functions have a simple pole. Denote this class by Σ.

Let $\Sigma^*(A,B)$, $\Sigma_K(A,B)$ $(-1\leqslant B<1, B<A)$ be the subclasses of functions in Σ satisfying

$$-\frac{zg'(z)}{g(z)}\prec\frac{1+Az}{1+Bz},\quad z\in U \tag{1.2}$$

and

$$-\left(\frac{zg''(z)}{g'(z)}+1\right)\prec\frac{1+Az}{1+Bz},\quad z\in U \tag{1.3}$$

respectively. Here $\prec$ denotes subordination and $U=\{z:|z|<1\}$. For $0\leqslant\beta<1$, $\Sigma^*(1-2\beta,-1)$ and $\Sigma_K(1-2\beta,-1)$ are respectively the well-known subclasses of Σ consisting of functions meromorphic starlike and meromorphic convex of order β. Denote by $\Sigma^*(1-2\beta,-1)=\Sigma^*(\beta)$ and $\Sigma^*(0)=\Sigma^*$; $\Sigma_K(1-2\beta,-1)=\Sigma_K(\beta)$ and $\Sigma_K(0)=\Sigma_K$. For more details about these classes see Chapter 17 of [3].

We say that a function $g\in\Sigma$ is meromorphic λ-spiral-like of order β in $0<|z|<1$ if

$$-\operatorname{Re}\left\{e^{i\lambda}\frac{zg'(z)}{g(z)}\right\}>\beta\cos\lambda,\quad z\in U \tag{1.4}$$

for $0\leqslant\beta<1$ and $-\pi/2<\lambda<\pi/2$. We denote this class by $\Sigma^{*\lambda}(\beta)$.

Similarly we say that function ψ of the form (1.1) analytic in $0<|z|<1$ is meromorphic close-to-convex of order β and type $\lambda (\beta < -1/d_{-1}, 0 \leqslant \lambda < 1)$ in $0<|z|<1$ if there exists a function ϕ of the form

$$\phi(z) = \frac{d_{-1}}{z} + \sum_{n=0}^{\infty} d_n z^n, \quad d_{-1} \neq 0, \tag{1.5}$$

which is meromorphic starlike of order λ (i.e. ϕ need not be normalized) such that

$$\operatorname{Re}\left\{\frac{z\psi'(z)}{\phi(z)}\right\} > \beta, \quad z \in U. \tag{1.6}$$

If we choose $\phi(z) = -1/z$, then the condition of meromorphic close-to-convexity reduces to

$$-\operatorname{Re}\{z^2 \psi'(z)\} > \beta, \quad z \in U. \tag{1.7}$$

Now let g and ψ be two functions with series expansions

$$g(z) = \frac{d_{-1}}{z} + \sum_{n=0}^{\infty} d_n z^n, \quad (d_{-1} \neq 0); \quad \psi(z) = \frac{e_{-1}}{z} + \sum_{n=0}^{\infty} e_n z^n, \quad (e_{-1} \neq 0)$$

which are analytic and univalent in $0<|z|<1$. In [7], Robertson showed that the convolution or Hadamard product $g*\psi$ of such functions defined by

$$(g*\psi)(z) = \frac{d_{-1}e_{-1}}{z} + \sum_{n=0}^{\infty} d_n e_n z^n$$

is also analytic and univalent in $0<|z|<1$, for $d_{-1} = e_{-1} = 1$. Furthermore he proved that $g*\psi$ is not only close-to-convex of order 0 in $0<|z|<1$ with respect to $\phi(z) = -1/z$ but also meromorphic starlike of order 0 in $0<|z|<1$.

Now we introduce the class $T_\delta(A,B)$ of functions having properties analogous to those of analytic univalent case considered by Ruscheweyh [8].

DEFINITION

Let δ, A, B be arbitrarily fixed real numbers such that $\delta > -1$, $-1 \leqslant B < 1$ with $B < A$. A function $g \in \Sigma$ is said to be in the class $T_\delta(A,B)$ if it satisfies

$$-\left[\frac{E^{\delta+1}g(z)}{E^\delta g(z)} - 2\right] \prec \frac{1+Az}{1+Bz}, \quad z \in U, \tag{1.8}$$

where

$$E^\delta g(z) = \frac{1}{z(1-z)^{\delta+1}} * g(z), \quad \delta > -1, \quad 0<|z|<1. \tag{1.9}$$

It is readily seen that for $\delta = n \in N \cup \{0\}$,

$$E^n g(z) = \frac{z^{-1}}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n}(z^{n+1} g(z)).$$

With this notation the well-known condition for $g\in\Sigma$ to be in $\Sigma^*(A,B)$ can be written as

$$g\in\Sigma^*(A,B) \text{ if and only if } g\in T_0(A,B).$$

Note that $E^0g(z)=g(z)$, $E^1g(z)=zg'(z)+2g(z)$ and so

$$E^1g(z)-2E^0g(z)=zg'(z)=\frac{2z-1}{z(1-z)^2}*g(z).$$

In order to prove the main theorems of this article, we need the following lemmas.

Lemma 1. [4, *Corollary* 3.2]. *If A, B, β are reals with $\beta\neq 0$, $-1\leqslant B<1$ and $A\neq B$, and complex number γ satisfies*

$$\operatorname{Re}\gamma\geqslant-\frac{\beta(1-A)}{2}\quad \textit{whenever } B=-1 \textit{ and } A\neq-1$$

and

$$\operatorname{Re}\gamma\geqslant\max\left\{-\frac{\beta(1-A)}{1-B},\ -\frac{\beta(1+A)}{1+B}\right\}\quad \textit{whenever } |B|<1 \textit{ and } A\neq B,$$

then the differential equation

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=\frac{1+Az}{1+Bz}$$

has a univalent solution given by

$$q(z)=\begin{cases}\dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B}\mathrm{d}t}-\dfrac{\gamma}{\beta} & \textit{if } B\neq 0\\[2ex] \dfrac{z^{\beta+\gamma}\exp(\beta Az)}{\beta\int_0^z t^{\beta+\gamma-1}\exp(\beta At)\mathrm{d}t}-\dfrac{\gamma}{\beta} & \textit{if } B=0.\end{cases}\tag{1.10}$$

If $p(z)$ is analytic in U and satisfies

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec\frac{1+Az}{1+Bz},\quad z\in U$$

then

$$p(z)\prec q(z)\prec\frac{1+Az}{1+Bz}$$

and $q(z)$ is the best dominant of the above differential subordination.

The method of proof of the following Lemma is based on the lines of proof of Wilken and Feng [10]. So we omit the details.

Lemma 2. *Let $\mu(t)$ be a positive measure on the unit interval $I=[0,1]$. Let $g(t,z)$ be a complex valued function defined on $[0,1]\times U$, analytic with respect to $z\in U$ and integrable in t for each $z\in U$ and for almost all $t\in[0,1]$, and suppose that* $\operatorname{Re}\{g(t,z)\}>0$

on U *and* $g(z) = \int_I g(t,z)\mathrm{d}\mu(t)$. *If, for fixed* $0 \leqslant \lambda < 2\pi$, $g(t, re^{i\lambda})$ *is real for* r *real and*

$$\mathrm{Re}\left\{\frac{1}{g(t,z)}\right\} \geqslant \frac{1}{g(t,re^{i\lambda})}, \quad \textit{for } |z| \leqslant r \textit{ and } t \in [0,1]$$

then

$$\mathrm{Re}\left\{\frac{1}{g(z)}\right\} \geqslant \frac{1}{g(re^{i\lambda})} \quad \textit{for } |z| \leqslant r \textit{ and } 0 \leqslant \lambda < 2\pi.$$

For a, b, c real numbers other than $0, -1, -2, \cdots$ the hypergeometric series

$$F(a,b;c;z) = 1 + \frac{a.b}{1.c}z + \frac{a(a+1)b(b+1)}{1.2c(c+1)}z^2 + \cdots \tag{1.11}$$

represents an analytic function in U [9, p. 281]. The following identities are well known.

Lemma 3. [9, *Chapter* XIV]: *For* a, b, c *real numbers other than* $0, -1, -2, \cdots$ *and* $c > b > 0$ *we have*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\mathrm{d}t = \frac{\Gamma(b).\Gamma(c-b)}{\Gamma(c)}F(a,b;c;z) \tag{1.12}$$

$$F(a,b;c;z) = F(b,a;c;z) \tag{1.13}$$

$$F(a,b;c;z) = (1-z)^{-a}F(a,c-b;c;z/(1-z)) \tag{1.14}$$

$$F\left(a,b;\frac{a+b+1}{2};\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}.$$

To a certain extent the work on univalent meromorphic functions has paralleled that of univalent analytic functions since one is tempted to search for a class of functions in Σ which is analogous to that of analytic case. However, though a large number of papers have appeared dealing with some of the classes defined, through convolution of analytic functions, it is somewhat surprising that not much attempt appears to have been made in defining the classes, through convolution of meromorphic functions. In the present paper we make a modest attempt in this direction. Recently, Mogra [5] has made an easy attempt in a different context.

In §2, we give a necessary and sufficient condition for a function g in Σ to be in $\Sigma^*(A,B)$ and $\Sigma_K(A,B)$, $(-1 \leqslant B < A \leqslant 1)$ respectively in terms of convolution. In §3, we give containment relations for the classes $T_{\delta+1}(A,B)$. In §4, we study certain integral transforms in the classes $T_\delta(A,B)$ and in turn to the classes $\Sigma^*(A,B)$ and $\Sigma_K(A,B)$ which are much more general than the one considered by Goel and Sohi [2], Bajpai [1] and others. In the last section we study some sort of converse problem for functions in $T_\delta(1-2\beta, -1)$. In the analytic case converse problem of this type has been considered by many workers (see [11]).

2. Convolution theorems

Theorem 2.1. *A function g is in $\Sigma^*(A,B)$, $-1 \leqslant B < A \leqslant 1$, if and only if*

$$\left[g(z)*\left\{\frac{1+(1-(A-2B)x)(A-B)^{-1}x^{-1}z}{z(1-z)^2}\right\}\right] \neq 0 \tag{2.1}$$

for $0<|z|<1$, $|x|=1$.

Proof. The function g is in $\Sigma^*(A,B)$, $-1 \leqslant B < A \leqslant 1$ if and only if

$$-\frac{zg'(z)}{g(z)} \neq \frac{1+Ax}{1+Bx} \tag{2.2}$$

for $z \in U$ and $|x|=1$. Since $-(zg'(z)/g(z))=1$ at $z=0$, (2.2) is equivalent to

$$-zg'(z)(1+Bx)-g(z)(1+Ax)\neq 0, \quad 0<|z|<1. \tag{2.3}$$

Since

$$g(z)=\frac{1}{z}+\sum_{n=0}^{\infty} b_n z^n = g(z)*\left(\frac{1}{z(1-z)}\right)$$

we have

$$\begin{aligned}-zg'(z)&=\frac{1}{z}-\sum_{n=0}^{\infty} nb_n z^n\\ &=g(z)*\left(\frac{1}{z}-\frac{z}{(1-z)^2}\right), \quad 0<|z|<1\\ &=g(z)*\left(\frac{1-2z}{z(1-z)^2}\right)\end{aligned}$$

so that the left hand side of (2.3) may be expressed as

$$g(z)*\left[\frac{1+(1-(A-2B)x)(A-B)^{-1}x^{-1}z}{z(1-z)^2}\right]\neq 0$$

for $0<|z|<1$, $|x|=1$, which is the desired convolution condition. This completes the proof. □

If we set

$$g(z)=\frac{1}{z}+\sum_{n=0}^{\infty} b_n z^n,$$

then

$$(zg'(z))'=(-zg'(z))*\left(\frac{1-2z}{z(1-z)^2}\right)$$

and so from Theorem 2.1, and the identity

$$zg'(z)*\psi(z)=g(z)*z\psi'(z), \quad g,\psi\in\Sigma$$

we obtain

Theorem 2.2. *A function g is in $\Sigma_K(A,B)$, $-1\leqslant B<A\leqslant 1$, if and only if*

$$.g(z)*\left[\frac{-1+(1-(A-2B)x)(A-B)^{-1}x^{-1}[3z+2z^2]}{z(1-z)^3}\right]\neq 0$$

for $0<|z|<1$, $|x|=1$.

Remark 1. For $A=1-2\beta, B=-1$, Theorems 2.1 and 2.2 give necessary and sufficient convolution conditions for a function $g\in\Sigma$ to be in $\Sigma^*(\beta)$ and $\Sigma_K(\beta)$, $0\leqslant\beta<1$ respectively.

Theorem 2.3. *A function $g\in\Sigma$ is λ-spiral-like of order β, $0\leqslant\beta<1$, in $0<|z|<1$ if and only if*

$$g(z)*\left[\frac{1+\left(\frac{2\beta\cos\lambda e^{-i\lambda}-e^{-2i\lambda}-2-x}{1+e^{-2i\lambda}-2\beta\cos\lambda e^{-i\lambda}}\right)z}{z(1-z)^2}\right]\neq 0 \tag{2.4}$$

for $0<|z|<1$, $|x|=1$.

Proof. The function $g\in\Sigma$ is λ-spiral-like of order β if and only if

$$-\operatorname{Re}\left\{e^{i\lambda}\frac{zg'(z)}{g(z)}\right\}>\beta\cos\lambda,\quad z\in U,\quad |\lambda|<\frac{\pi}{2}. \tag{2.5}$$

Since

$$\left(-e^{i\lambda}\frac{zg'(z)}{g(z)}-i\sin\lambda\right)(\cos\lambda)^{-1}=1 \text{ at } z=0,$$

(2.5) is equivalent to

$$\left(-e^{i\lambda}\frac{zg'(z)}{g(z)}-i\sin\lambda\right)(\cos\lambda)^{-1}\neq\frac{1+(2\beta-1)x}{1+x},\quad z\in U, |x|=1, x\neq -1$$

which simplifies to

$$-(1+\bar{x})zg'(z)+(e^{-2i\lambda}-2\beta\cos\lambda e^{-i\lambda}-\bar{x})g(z)\neq 0, \tag{2.6}$$

where $\bar{x}$ stands for the conjugate of the complex x. The remainder of the argument is the same as that of Theorem 2.1. □

3. Containment relations

For the proof of our next theorem we need the following lemma:

Lemma 3.1. *For $g\in\Sigma$ and $\delta>-1$ we have the following identity*

$$z\frac{\mathrm{d}}{\mathrm{d}z}(E^\delta g(z))=(\delta+1)E^{\delta+1}g(z)-(\delta+2)E^\delta g(z),\quad 0<|z|<1 \tag{3.1}$$

where $E^\delta g(z)$ is given by (1.9).

Proof. If we set

$$g(z)=\frac{1}{z}+\sum_{n=0}^{\infty} b_n z^n,$$

then

$$\begin{aligned} E^{\delta}g(z) &= \frac{1}{z(1-z)^{\delta+1}} * g(z) \\ &\equiv \left[\frac{1}{z}+\sum_{n=0}^{\infty}\binom{\delta+n+1}{n+1}z^n\right] * \left[\frac{1}{z}+\sum_{n=0}^{\infty} b_n z^n\right]. \end{aligned} \tag{3.2}$$

Now equating the coefficients of z^{-1}, constant term and z^n in the expansion of $(\delta+1)E^{\delta+1}g(z)-(\delta+2)E^{\delta}g(z)$ with the corresponding coefficients in the expansion of $z(E^{\delta}g(z))'$, the result follows. □

Theorem 3.1. *Let* $\delta > -1$.

(a) *Suppose that the constants A and B satisfy*

$$B<A\leqslant\frac{\delta+1+B}{\delta+2} \quad \textit{for } -1\leqslant B<1. \tag{3.3}$$

Then for $g\in T_{\delta+1}(A,B)$ *we have*

$$g\in T_{\delta}(A',B) \tag{3.4}$$

where

$$A'=A+(A-B)/(\delta+1). \tag{3.5}$$

Further

$$-\left(\frac{E^{\delta+1}g(z)}{E^{\delta}g(z)}-2\right) \prec -\frac{1}{\delta+1}\left[\frac{1}{Q(z)}\right]+2\equiv\tilde{q}(z)\prec\frac{1+A'z}{1+Bz}, \quad z\in U \tag{3.6}$$

where

$$Q(z)=\begin{cases}\displaystyle\int_0^1\left(\frac{1+Btz}{1+Bz}\right)^{-(\delta+2)(A-B)/B} t^{\delta}\mathrm{d}t, & \textit{if } B\neq 0\\[2ex] \displaystyle\int_0^1 \exp\{(\delta+2)(1-t)Az\}t^{\delta}\mathrm{d}t, & \textit{if } B=0.\end{cases} \tag{3.7}$$

(b) *Suppose that the constants A and B satisfy*

$$B<A\leqslant\min\left\{\frac{\delta+1+B}{\delta+2},2B\right\}$$

with $0<B<1$, *then*

$$T_{\delta+1}(A,B)\subset T_{\delta}(1-2\rho',-1) \tag{3.8}$$

where

$$\rho'=2-[F(1,(\delta+2)(A-B)/B;\delta+2;B/(1+B))]^{-1}. \tag{3.9}$$

(c) *Suppose that*

$$B<A\leqslant\min\left\{\frac{\delta+1+B}{\delta+2},0\right\}$$

with $-1 \leqslant B < 0$, *then*

$$T_{\delta+1}(A,B) \subset T_\delta(1-2\rho'', -1) \tag{3.10}$$

where

$$\rho'' = 2 - [F(1, (\delta+2)(B-A)/B; \delta+2; -B/(1-B))]^{-1} \tag{3.11}$$

and $F(a,b;c;z)$ *is as defined by* (1.11). *The relations* (3.6), (3.8) *and* (3.10) *are all the best possible.*

Proof. Let $g \in T_{\delta+1}(A,B)$ where $\delta > -1$, $-1 \leqslant B < 1$ and $B < A$. Set

$$\phi(z) = z[zE^\delta g(z)]^{-1/(1+\delta)}$$

and

$$r_1 = \sup\{r : \phi(z) \neq 0, \quad 0 < |z| < r < 1\}.$$

Then ϕ is single valued in $0 < |z| < r_1$ and using (3.1) it follows that the function p defined by

$$p(z) = \frac{z\phi'(z)}{\phi(z)} = -\left(\frac{E^{\delta+1}g(z)}{E^\delta g(z)} - 2\right) \tag{3.12}$$

is analytic in $|z| < r_1$ and $p(0) = 1$. Since $g \in T_{\delta+1}(A,B)$, (1.8) coupled with (3.12) and (3.1) easily leads to

$$-\left(\frac{E^{\delta+2}g(z)}{E^{\delta+1}g(z)} - 2\right) = \frac{1}{\delta+2} + \frac{\delta+1}{\delta+2}\left(p(z) + \frac{zp'(z)}{(\delta+1)(2-p(z))}\right)$$
$$\prec \frac{1+Az}{1+Bz}, \quad |z| < r_1.$$

In other words,

$$P(z) + \frac{zP'(z)}{\beta P(z) + \gamma} \prec \frac{1+Az}{1+Bz} \tag{3.13}$$

where

$$P(z) = \left(1 - \frac{1}{\delta+2}\right)p(z) + \frac{1}{\delta+2}, \quad \beta = -(\delta+2), \quad \gamma = 2\delta+3. \tag{3.14}$$

It can be easily seen that for $-1 \leqslant B < 1$ and $A \neq B$,

$$\operatorname{Re}\left\{\beta\left(\frac{1+Az}{1+Bz}\right) + \gamma\right\} > 0 \text{ in } U$$

if and only if A and B satisfy the following inequalities:

$$\frac{-(\delta+1) + (2\delta+3)B}{\delta+2} \leqslant A \leqslant \frac{(\delta+1)+(2\delta+3)B}{\delta+2} \quad \text{for } -1 < B < 1$$

and

$$A \geqslant -\frac{(3\delta+4)}{\delta+2} \quad \text{for } B = -1.$$

It may be noted that (3.3) obviously satisfies the above inequalities and so it follows that

$$\operatorname{Re}\left\{\beta\left(\frac{1+Az}{1+Bz}\right)+\gamma\right\}>0 \text{ in } U$$

under the condition (3.3). Now using Lemma 1 we deduce that

$$P(z)\prec q(z)\prec\frac{1+Az}{1+Bz},\quad |z|<r_1 \tag{3.15}$$

where q is the best dominant of (3.13) and is given by

$$q(z)=\frac{\delta+1}{\delta+2}\tilde{q}(z)+\frac{1}{\delta+2}.$$

Again by (3.14) and (3.15) we easily get

$$p(z)\prec\tilde{q}(z)\equiv\frac{-1}{(\delta+1)}\left[\frac{1}{Q(z)}\right]+2\prec\frac{1+((A(\delta+2)-B)/(\delta+1))z}{1+Bz},\quad |z|<r_1 \tag{3.16}$$

where $Q(z)$ is given by (3.7). From (3.3) and (3.16) we see that Re $p(z)>0$ in $|z|<r_1$. This, by (3.12), shows that ϕ is starlike (univalent) in $|z|<r_1$. Thus it is not possible that ϕ vanishes in $|z|<r_1$ if $r_1<1$. So we conclude that $r_1=1$. Therefore p is analytic in U. Hence by (3.12) and (3.16),

$$g\in T_{\delta+1}(A,B) \text{ implies } -\left(\frac{E^{\delta+1}g(z)}{E^{\delta}g(z)}-2\right)\prec\tilde{q}(z),\quad z\in U$$

provided δ, A and B satisfy (3.3). This proves (3.4) and (3.6).

(b) Next we show that

$$\inf_{|z|<1}\{\operatorname{Re}\tilde{q}(z)\}=\tilde{q}(1),\quad \tilde{q}(z)=2-\frac{1}{\delta+1}\left[\frac{1}{Q(z)}\right] \tag{3.17}$$

provided $\delta>-1$, A and B satisfy

$$B<A<\min\left\{\frac{\delta+1+B}{\delta+2},2B\right\}. \tag{3.18}$$

If we set

$$a=-\beta\left(\frac{A-B}{B}\right),\quad b=\beta+\gamma,\quad c=\beta+\gamma+1(\beta=-(\delta+2),\gamma=2\delta+3)$$

then $c>b>0$. From (3.7) by using (1.12), (1.13) and (1.14) we see that for $0\neq B$

$$Q(z)=(1+Bz)^a\int_0^1(1+Btz)^{-a}t^{b-1}\mathrm{d}t$$

$$= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a, c-b; c; Bz/(1+Bz))$$

$$= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(1, a; c; Bz/(1+Bz)). \tag{3.19}$$

Again (3.7), by (3.19) for $0 < B < 1$, $B < A < \min\left\{\frac{\delta+1+B}{\delta+2}, 2B\right\}$, so that $c > a > 0$, can be rewritten as

$$Q(z) = \int_0^1 g(t,z)\mathrm{d}\mu(t)$$

with

$$g(t,z) = \frac{1+Bz}{1+(1-t)Bz}$$

and

$$\mathrm{d}\mu(t) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)} t^{a-1}(1-t)^{c-a-1}\mathrm{d}t.$$

Using Lemma 2, (with $\lambda = 0$) we easily obtain

$$\mathrm{Re}\left\{\frac{1}{Q(z)}\right\} \geqslant \frac{1}{Q(1)}, \quad z \in U.$$

This proves (3.17) and so by (3.6) we obtain (3.8). For the case $A = \min\left\{\frac{\delta+1+B}{\delta+2}, 2B\right\}$ we obtain (3.8) by letting $A \to \left[\min\left\{\frac{\delta+1+B}{\delta+2}, 2B\right\}\right]^+$.

(c) In a manner similar to that of part (b), using the Lemma 2 (with $\lambda = \pi$) we can easily show

$$\inf_{|z|<1} \mathrm{Re}\,\tilde{q}(z) = \tilde{q}(-1)$$

provided $\delta > -1$, A and B satisfy $-1 \leqslant B < 0$ with $B < A \leqslant \min\left\{0, \frac{(\delta+1)+B}{\delta+2}\right\}$. The proof of part (c) now follows on the same lines. Sharpness follows from the best dominant property. □

4. Integral transforms

We study in this section certain integral transforms of functions in the class $T_\delta(A,B)$. For a function $g \in \Sigma$, defined by $g(z) = z^{-1} + \Sigma_{n=0}^{\infty} b_n z^n$, Bajpai [2] defined the integral transform G_c by

$$G_c(z) = \begin{cases} c\displaystyle\int_0^1 u^c g(uz)\mathrm{d}u \\ z^{-1} + \displaystyle\sum_{n=0}^{\infty} \frac{c}{c+n+1} b_n z^n \end{cases} \tag{4.1}$$

where c is real with $c \geqslant 1$ and showed that

$$g \in \Sigma^*(\beta) \text{ implies } G_c \in \Sigma^*(\beta), \quad (0 \leqslant \beta < 1). \tag{4.2}$$

Reddy and Juneja [6] showed that the relation (4.2) continues to hold if c in (4.1) is taken to be a complex number satisfying $\operatorname{Re} c > 0$. They improved the relation (4.2) further by using a weaker condition on g. In the following theorem we give the sharp results in generalized form as follows.

Theorem 4.1. *Let $\delta > -1$, c be a complex number satisfying $\operatorname{Re} c > 0$ and the constants A, B, δ and c satisfy*

$$B - \frac{(1-B)\operatorname{Re} c}{\delta+1} \leqslant A \leqslant B + \frac{(1+B)\operatorname{Re} c}{\delta+1} \quad \text{for } -1 < B < 1 \tag{4.3}$$

and

$$A \geqslant -1 - \frac{2\operatorname{Re} c}{\delta+1} \quad \text{for } B = -1. \tag{4.4}$$

(a) *If $g \in T_\delta(A,B)$ then the function G_c defined by (4.1) satisfies $G_c \in T_\delta(A,B)$. Furthermore*

$$-\left[\frac{E^{\delta+1}G_c(z)}{E^\delta G_c(z)} - 2\right] \prec -\frac{1}{\delta+1}\left[\frac{1}{Q(z)}\right] + \left(\frac{c+\delta+1}{\delta+1}\right) \equiv \tilde{q}(z), \quad z \in U \tag{4.5}$$

where

$$Q(z) = \begin{cases} \displaystyle\int_0^1 \left(\frac{1+Btz}{1+Bz}\right)^{-(\delta+1)(A-B)/B} t^{c-1}\,\mathrm{d}t & \text{if } B \neq 0 \\ \displaystyle\int_0^1 \exp\{-(1+\delta)A(t-1)z\}\,t^{c-1}\,\mathrm{d}t & \text{if } B = 0. \end{cases} \tag{4.6}$$

(b) *If c is real with $c > 0$, $0 < B < 1$ and $B < A \leqslant \min\left\{B + \dfrac{(c+1)B}{\delta+1}, B + \dfrac{(1+B)c}{\delta+1}\right\}$ then for $g \in T_\delta(A,B)$ we have*

$$G_c \in T_\delta(1-2\rho', -1) \tag{4.7}$$

where

$$\rho' = \frac{1}{1+\delta}\{c+\delta+1 - [F(1,(\delta+1)(A-B)/B; c+1; B/(1+B))]^{-1}. \tag{4.8}$$

(c) *If c is real with $c > 0$,*

$$B < A \leqslant \min\left\{-\frac{B(c-\delta)}{\delta+1}, B + \frac{(1+B)c}{\delta+1}\right\} \quad \text{for } -1 < B < 0$$

and

$$-1 < A \leqslant \frac{c-\delta}{\delta+1} \quad \text{for } B = -1$$

then for $g \in T_\delta(A,B)$ we have

$$G_c \in T_\delta(1-2\rho'', -1) \tag{4.9}$$

where

$$\rho'' = \frac{1}{\delta+1}\{c+\delta+1-[F(1,(\delta+1)(B-A)/B;c+1;-B/(1-B))]^{-1}\}. \tag{4.10}$$

The results are all the best possible ones.

Proof. Suppose that $g \in T_\delta(A,B)$ and A, B, δ and c satisfy (4.3) and (4.4). Since G_c is defined by

$$G(z) \equiv G_c(z) = \left(z^{-1} + \sum_{n=0}^{\infty} \frac{c}{c+n+1} z^n\right) * g(z)$$

and

$$E^\delta g(z) = z^{-1} + \sum_{n=0}^{\infty} \frac{\Gamma(n+\delta+2)}{\Gamma(\delta+1)} \frac{1}{(n+1)!} z^n$$

it can be easily seen that

$$z\frac{\mathrm{d}}{\mathrm{d}z}[E^\delta G(z)] = cE^\delta g(z) - (1+c)E^\delta G(z) \tag{4.11}$$

and as in Lemma 3.1,

$$z\frac{\mathrm{d}}{\mathrm{d}z}[E^\delta G(z)] = (\delta+1)E^{\delta+1}G(z) - (\delta+2)E^\delta G(z). \tag{4.12}$$

Equating (4.11) and (4.12) we get

$$cE^\delta g(z) = (c-\delta-1)E^\delta G(z) + (\delta+1)E^{\delta+1}G(z). \tag{4.13}$$

Let $g \in T_\delta(A,B)$. We put

$$\phi(z) = z[zE^\delta G(z)]^{-1/(\delta+1)}$$

and $r_1 = \sup\{r: \phi(z) \neq 0, 0 < |z| < r < 1\}$. Then ϕ is single valued and analytic in $|z| < r_1$ and p defined by

$$p(z) = \frac{z\phi'(z)}{\phi(z)} = -\left(\frac{E^{\delta+1}G(z)}{E^\delta G(z)} - 2\right) \tag{4.14}$$

is analytic in $|z| < r_1$, $p(0) = 1$.

Using (4.11) and (4.12), (4.14) by differentiation reduces to

$$-\left(\frac{E^{\delta+1}g(z)}{E^\delta g(z)} - 2\right) = p(z) + \frac{zp'(z)}{c+\delta+1-(\delta+1)p(z)}, \quad |z| < r_1. \tag{4.15}$$

Since $g \in T_\delta(A,B)$, we have by (4.15) that

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1+Az}{1+Bz}, \tag{4.16}$$

where

$$\beta = -(\delta+1), \quad \gamma = c+\delta+1.$$

Using Lemma 1, we deduce that

$$p(z) \prec \tilde{q}(z) = \frac{1}{\beta}\left[\frac{1}{Q(z)}\right] - \frac{\gamma}{\beta} \prec \frac{1+Az}{1+Bz}, \quad |z| < r_1, \tag{4.17}$$

where $Q(z)$ is defined by (4.6). It may be noted that for $\beta = -(\delta+1)$ and $\gamma = c+\delta+1$ with $-1 \leqslant B < 1$ and $B < A$,

$$\operatorname{Re}\left\{\beta\left(\frac{1+Az}{1+Bz}\right) + \gamma\right\} > 0 \text{ in } U$$

if the conditions (4.3) and (4.4) are satisfied. Thus it is not possible that $\phi(z)$ vanishes in $|z| < r_1$ if $r_1 < 1$. So we conclude that $r_1 = 1$. Therefore p is analytic in U and hence by (4.14) and (4.17) we obtain the first part of the theorem from Lemma 1.

(b) For the second part it is enough to show that

$$\inf_{|z|<1} \tilde{q}(z) = \tilde{q}(1), \tag{4.18}$$

provided $\delta > -1$, c, A and B satisfy $0 < B < 1$ with

$$A < B \leqslant \min\left\{B + \frac{(c+1)B}{\delta+1}, B + \frac{(1+B)c}{\delta+1}\right\},$$

where

$$\tilde{q}(z) = \frac{-1}{\delta+1}\left[\frac{1}{Q(z)}\right] + \frac{c+\delta+1}{\delta+1}$$

and $Q(z)$ is given by (4.6). If we set

$$a = -\beta(A-B)/B, b = \beta+\gamma, c' = \beta+\gamma+1, \quad (\beta = -(\delta+1), \gamma = c+\delta+1)$$

then $c' > b > 0$. From (4.6) we as before see that for $B \neq 0$

$$\begin{aligned} Q(z) &= (1+Bz)^a \int_0^1 (1+Btz)^{-a} t^{b-1}\,\mathrm{d}t \\ &= \frac{\Gamma(b)\Gamma(c'-b)}{\Gamma(c')} F(1, a; c'; Bz/(1+Bz)). \end{aligned}$$

For $0 < B < 1, B < A < \min\left\{B + \dfrac{(c+1)B}{1+\delta}, B + \dfrac{(1+B)c}{1+\delta}\right\}$ we see that $c' > a > 0$ and so (4.6) can be rewritten as

$$Q(z) = \int_0^1 g(t,z)\,\mathrm{d}\mu(t)$$

where

$$g(t,z) = \frac{1+Bz}{1+(1-t)Bz},$$

and

$$\mathrm{d}\mu(t) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(c'-a)} t^{a-1}(1-t)^{c'-a-1}\,\mathrm{d}t.$$

Using Lemma 2, (with $\lambda = 0$), we easily obtain

$$\mathrm{Re}\left\{\frac{1}{Q(z)}\right\} \geqslant \frac{1}{Q(1)}, \quad z \in U.$$

This proves (4.18) and hence by (4.17) and (4.14) we obtain (4.7). For the case

$$A = \min\left\{B + \frac{(c+1)B}{1+\delta}, B + \frac{(1+B)c}{1+\delta}\right\}.$$

we obtain (4.7) by taking limit as in Theorem 3.1(b).

Part (c) can be proved on similar lines using Lemma 2 with $\lambda = \pi$.

Taking $\delta = 0$ in the above theorem we obtain

COROLLARY 4.1.

Let c be a complex number satisfying $\mathrm{Re}\, c > 0$, and such that

$$B - (1-B)\mathrm{Re}\, c < A \leqslant B + (1+B)\mathrm{Re}\, c \text{ when } -1 < B < 1,$$

and

$$A \geqslant -1 - 2\mathrm{Re}\, c \text{ when } B = -1.$$

(a) *If $g \in \Sigma^*(A, B)$ then the function G_c defined by* (4.1) *satisfies*

$$-\mathrm{Re}\left(\frac{zG_c'(z)}{G_c(z)}\right) > \inf \mathrm{Re}\left[-\left(\frac{1}{Q(z)}\right) + \frac{c+\delta+1}{\delta+2}\right]$$

where

$$Q(z) = \begin{cases} \displaystyle\int_0^1 \left(\frac{1+Btz}{1+Bz}\right)^{(B-A)/B} t^{c-1}\,\mathrm{d}t & \text{if } B \neq 0 \\ \displaystyle\int_0^1 \exp\{A(1-t)z\} t^{c-1}\,\mathrm{d}t & \text{if } B = 0. \end{cases}$$

(b) *If c is real with $c > 0, 0 < B < 1$ and $B < A \leqslant \min\{B + (1+c)B, B + (1+B)c\}$ t*

$$g \in \Sigma^*(A, B) \text{ implies } G_c \in \Sigma^*(1 - 2\rho_1', -1)$$

where

$$\rho_1' = c + 1 - [F(1, (A-B)/B; c+1; B/(1+B))]^{-1}.$$

(c) *If c is real with $c > 0$,*

$$B < A \leqslant \min\{-Bc, B + (1+B)c\} \text{ when } -1 < B < 0$$

and

$$-1 < A \leqslant c \text{ when } B = -1$$

then

$$g \in \Sigma^*(A, B) \text{ implies } G_c \in \Sigma^*(1 - \rho_2'', -1)$$

where

$$\rho_2'' = c + 1 - [F(1, (B-A)/B; c+1; -B/(1-B))]^{-1}.$$

Remark 2. Since $g \in \Sigma_K(A,B)$ if and only if $-zg'(z) \in \Sigma^*(A,B)$ one can show that the Corollary 4.2 remains true on replacing $\Sigma^*(A,B)$ by $\Sigma_K(A,B)$ both in the hypothesis and the conclusion of the theorem.

Substituting $A = 1 - 2\rho$ with $B = -1$ in part (c) of the above theorem, we get the following sharp result.

COROLLARY 4.2.

Let $g \in \Sigma$, *and for real* $c > 0$, *let*

$$G_c(z) = \left(z^{-1} + \sum_{n=0}^{\infty} \frac{c}{c+n+1} z^n \right) * g(z).$$

Then for $-(c-1)/2 \leqslant \rho < 1$ *we have*

$$g \in \Sigma^*(\rho) \text{ (or } g \in \Sigma^k(\rho) \text{ resp.) implies } G_c \in \Sigma^*(\rho_3'') \text{ (or } G_c \in \Sigma^k(\rho_3'') \text{ resp.)} \tag{4.19}$$

where

$$\rho_3'' = c + 1 - [F(1,(1-\rho)/2; c+1; 1/2)]^{-1}.$$

The result is sharp.

The above Corollary shows that the result obtained improves (and also extends) the earlier results of Bajpai [1], Goel and Sohi [2], Reddy and Juneja [6] and others.

5. Inverse problem

In the earlier section we considered the function G_c defined by (4.1). If we want to solve the equation for g in terms of G_c we must first see that

$$g(z) = \frac{1}{c}[(1+c)G_c(z) + zG_c'(z)]. \tag{5.1}$$

Given some property of $G_c(z)$, we can ask questions about the nature of $g(z)$. A problem of this type can be regarded as a sort of inverse problem.

For considering such an inverse problem of Theorem 4.1 for $A = 1 - 2\rho$ and $B = -1$ we need the following, proof of which can be had in [11]. However we only need the special case of it in this article.

Theorem 5.1. *Let p and q be regular in U and* $\operatorname{Re} p(z) > 0$, $q(0) = 1$ *and* $\operatorname{Re} q(z) > 0$ *for* $z \in U$. *Further let $C \neq 0$ and D be complex constants such that $C + D \neq 0$. Then*

$$\operatorname{Re}\left\{ p(z) + \frac{zp'(z)}{C + Dq(z)} \right\} > 0 \text{ in } |z| < \rho(C,D) \tag{5.2}$$

where

$$\rho(C,D) = \frac{[|C|^2 + 2 + 4|D| + |D|^2 - \sqrt{R}]^{1/2}}{|C-D|} \tag{5.3}$$

$$= \frac{|C+D|}{[|C|^2 + 2 + 4|D| + |D|^2 + \sqrt{R}]^{1/2}}$$

and

$$R = (|C|^2 + 2 + 4|D| + |D|^2)^2 - |C^2 - D^2|^2.$$

The result is sharp for C real and non-negative constant.

In [2], Goel and Sohi proved that if G_c belongs to Σ^* then g defined by (5.1) is meromorphically star-like in $0 < |z| < [c/(c+2)]^{1/2}$. However their claim is incorrect. This can be easily seen from the following example.

Consider

$$G(z) = \frac{(1+z)^2}{z} = \frac{1}{z} + 2 + z \text{ with } c = 2/3.$$

Then from (5.1) we have

$$g(z) = \frac{1}{z} + 5 + 4z,$$

and hence

$$-\frac{zg'(z)}{g(z)} = \frac{1-4z^2}{1+5z+4z^2} = \frac{1-4z^2}{(4z+1)(z+1)}.$$

Therefore according to Goel and Sohi [2] g is meromorphically star-like in $0 < |z| < 1/2$. But this is not so because

$$-\operatorname{Re}\frac{zg'(z)}{g(z)}\bigg|_{z=-1/3} = -\frac{5}{2} < 0.$$

The following theorem not only rectifies the result of Goel and Sohi [2] but also generalizes the corresponding results in this direction.

Theorem 5.2. *Let $G_c \in T_\delta(\beta)$, $(0 \leqslant \beta < 1)$ and c be a complex number such that* $\operatorname{Re} c > 0$. *Define g by*

$$g(z) = \frac{1}{c}[(1+c)G_c(z) + zG_c'(z)]. \tag{5.4}$$

Then g belongs to $T_\delta(\beta)$ in $|z| < \rho(C, D)$ for $C = c + (1+\delta)(1-\beta)$ and $D = -(1+\delta)(1-\beta)$ where $\rho(C, D)$ is given by (5.3).

Proof. Since $G_c \in T_\delta(\beta)$ we can write

$$p(z) = (1-\beta)^{-1}\left[-\left(\frac{E^{\delta+1}G_c(z)}{E^\delta G_c(z)} - 2\right) - \beta\right]. \tag{5.5}$$

By hypothesis p is analytic, $\operatorname{Re} p(z) > 0$ in U and $p(0) = 1$. As in the proof of Theorem 4.1 we, from (5.4) and (5.5), easily get the identity

$$(1-\beta)^{-1}\left[-\left(\frac{E^{\delta+1}g(z)}{E^\delta g(z)} - 2\right) - \beta\right] = p(z) + \frac{zp'(z)}{c + (\delta+1)(1-\beta) - (1+\delta)(1-\beta)p(z)}.$$

Set

$$C = c + (1+\delta)(1-\beta) \text{ and } D = -(1+\delta)(1-\beta) \tag{5.6}$$

so that $C + D \neq 0$. Hence by Theorem 5.1 we obtain that $g \in T_\delta(\beta)$ in $|z| < \rho(C, D)$ where $\rho(C, D)$ is obtained from (5.3) using (5.6). ☐

Substituting $\delta = 0$ in the above theorem and noting Remark 2 we obtain the rectified form of [2] in the following corollary.

COROLLARY 5.1.

If G_c belongs to $\Sigma^*(\beta)$ (or $\Sigma_K(\beta)$ resp.) for $0 \leqslant \beta < 1$ and

$$g(z) = \frac{1}{c}[(1+c)G_c(z) + zG_c'(z)], \quad \mathrm{Re}\, c > 0$$

then

$$g \in \Sigma^*(\beta) \text{ (or } \Sigma_K(\beta) \text{ resp.) in } |z| < \rho(C, D)$$

with

$$C = c + (1-\beta) \text{ and } D = -(1-\beta).$$

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Simultaneous operational calculus involving a product of a general class of polynomials, Fox's *H*-function and the multivariable *H*-function

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Abstract. New operational relations between the original and the image for two-dimensional Laplace transforms involving a general class of polynomials, Fox's *H*-function and the multivariable *H*-function are obtained. The result provides a unification of the bivariate Laplace transforms for the *H*-functions given by Chaurasia [2, 3].

Keywords. Laplace transform; Fox's *H*-function; multivariable *H*-function; general class of polynomials; original and image functions; operational calculus; integral equation.

1. Introduction

The Laplace-Carson transform in two variables is defined and represented by the integral equation [4, p. 39]

$$F(p,q) = pq\int_0^\infty\int_0^\infty \exp(-px-qy)f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \quad \mathrm{Re}(p,q)>0, \tag{1}$$

where $F(p,q)$ and $f(x,y)$ are said to be operationally related to each other. $F(p,q)$ is called the image and $f(x,y)$ the original. Symbolically, we can write

$$F(p,q) \doteqdot f(x,y) \text{ or vice versa} \tag{2}$$

and the symbol $\doteqdot$ is called operational.

Srivastava [7, p. 1, eqn. (1)] has introduced the general class of polynomials

$$S_\lambda^\mu[x] = \sum_{\alpha=0}^{[\lambda/\mu]} \frac{(-\lambda)_{\mu\alpha}}{\alpha!} L_{\lambda,\alpha} x^\alpha, \quad \lambda = 0,1,2,\ldots, \tag{3}$$

where μ is an arbitrary positive integer and the coefficients $L_{\lambda,\alpha}(\lambda,\alpha \geqslant 0)$ are arbitrary constants real or complex. On suitably specializing the coefficients $L_{\lambda,\alpha}$, $S_\lambda^\mu[x]$ yields a number of known polynomials as its particular cases. These include, among others, Hermite polynomials, Jacobi polynomials, Laguerre polynomials, Bessel polynomials, Gould-Hopper polynomials, Brafman polynomials and several others [9, pp. 158–61].

The series representation of Fox's *H*-function [1, 5] is

$$H_{u,d}^{m,n}\left[z\,\middle|\,\begin{matrix}(e_a,E_a)\\(f_d,F_d)\end{matrix}\right] = \sum_{G=1}^{m}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\,F_G}\phi(g_s)z^{g_s} \tag{4}$$

where

$$\phi(g_s) = \prod_{j=1, j\neq G}^{m} \Gamma(f_j - F_j g_s) \prod_{j=1}^{n} \Gamma(1 - e_j + E_j g_s) \times \left\{ \prod_{j=1+m}^{d} \Gamma(1 - f_g + F_j g_s) \prod_{j=1}^{u} \Gamma(a_j - E_j g_s) \right\}^{-1} \tag{5}$$

and $g_s = (f_G + s)/F_G$.

Srivastava and Panda [8] have introduced the multivariable H-function as

$$H^{0,u:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}\left(\begin{matrix}[(a): A',\ldots,A^{(r)}]:[(b'):B'];\ldots;[(b^{(r)}):B^{(r)}];\\ [(c): C',\ldots,C^{(r)}]:[(d'):D'];\ldots;[(d^{(r)}):D^{(r)}];\end{matrix} z_1,\ldots,z_r\right). \tag{6}$$

The defining integral of the above function, its various special cases and other details can be found in the paper referred to above.

For the sake of brevity

$$T_i = -\sum_{j=u+1}^{u} A_j^{(i)} + \sum_{j=1}^{N^{(i)}} B_j^{(i)} - \sum_{j=1+N^{(i)}}^{P^{(i)}} B_j^{(i)} - \sum_{j=1}^{w} C_j^{(i)} + \sum_{j=1}^{M^{(i)}} D_j^{(i)} - \sum_{j=1+M^{(i)}}^{Q^{(i)}} D_j^{(i)} \tag{7}$$

$$\left.\begin{aligned} &\theta_i = d_j^{(i)}/D_j^{(i)}, \quad j = 1,\ldots,M^{(i)} \\ &\phi_i = (1 - b_j^{(i)})/B_j^{(i)}, \quad j = 1,\ldots,N^{(i)}, \quad i = 1,\ldots,r \end{aligned}\right\} \tag{8}$$

$$T = \sum_1^n E_i - \sum_{n+1}^{u} E_i + \sum_1^m F_i - \sum_{m+1}^{d} F_i \tag{9}$$

$$\left.\begin{aligned} &\theta = f_l/F_l, \quad l = 1,\ldots,m \\ &\phi = (e_{l'} - 1)/E_{l'}, \quad l' = 1,\ldots,n \end{aligned}\right\}. \tag{10}$$

Also we use the notation

$$H^{0,u:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}(z_1,\ldots,z_r) \tag{11}$$

to denote (6).

The importance of our main result lies in the fact that it involves the product of a general class of polynomials, Fox's H-function and the multivariable H-function having general arguments. Thus this result serves as a key formula from which the bivariate Laplace transform for the product of a large number of polynomials which are special cases of $S_\lambda^\mu[x]$ [9, pp. 158–61] and simpler special functions which are particular cases of Fox's H-function [6, pp. 145–151] and of the multivariable H-function follow merely by specializing the parameters. In this paper we shall obtain correspondences, involving a product of a general class of polynomials, Fox's H-function and the multivariable H-function, between the original and the image in two variables.

In what follows we shall denote the original variables by x and y and the transformed variables by p and q. The notations employed are those of Ditkin and Prudnikov's operational calculus.

Theorem 1. *With* T_i, θ_i, ϕ_i, T, θ *and* ϕ *given by* (7), (8), (9) *and* (10) *respectively, let* $T_i > 0$, $T > 0$, $|\arg(z_i)| < T_i\pi/2$, $|\arg(z)| < T\pi/2$, $h_i > 0$, $k > 0$, $\delta > 0$, $i = 1, \ldots, r$, μ *is an arbitrary positive integer and the coefficients* $L_{\lambda,\alpha}(\lambda, \alpha \geqslant 0)$ *are arbitrary constants real or complex, and*

(i) $\mathrm{Re}\left(\rho - \sigma - k\phi - \sum_{i=1}^{r} h_i\phi_i\right) < 3/4$

(ii) $\mathrm{Re}(\rho) > 0, \quad \mathrm{Re}\left(\sigma + k\theta + \sum_{i=1}^{r} h_i\theta_i\right) > 0,$

Also, let $0 \leqslant n \leqslant u$, $0 \leqslant m \leqslant d$ *and*

(iii) $R(\rho) > 0$.

$$\begin{aligned}
&p^{-1/2}(pq)^{\sigma/2-\rho+1}\sum_{\alpha=0}^{[\lambda/\mu]}\sum_{G=1}^{m}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!F_G}\phi(g_s)\\
&\times z^{g_s}\frac{(-\lambda)_{\mu\alpha}}{\alpha!}L_{\lambda,\alpha}(pq)^{(kg_s+\delta\alpha/2}\cdot H^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}(z_1(\sqrt{pq})^{h_1},\ldots,z_r(\sqrt{pq})^{h_r})\\
&\doteqdot\frac{(4xy)^{\rho-(\sigma/2)-(1/2)}}{\sqrt{\pi y}}\sum_{\alpha=0}^{[\lambda/\mu]}\sum_{G=1}^{m}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!F_G}\phi(g_s)z^{g_s}\frac{(-\lambda)_{\mu\alpha}}{\alpha!}\\
&\times L_{\lambda,\alpha}(4xy)^{(-kg_s/2)-(\delta\alpha/2)}H^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v+1,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}\left(\begin{matrix}[(a):A',\ldots,A^{(r)}],\\ [(c):C',\ldots,C^{(r)}]:\end{matrix}\right.\\
&\times[2\rho-\sigma-kg_s-\delta\alpha:h_1,\ldots,h_r]:\begin{matrix}[(b'):B'];\ldots;[(b^{(r)}):B^{(r)}];\\ [(d'):D'];\ldots;[(d^{(r)}):D^{(r)}];\end{matrix}\\
&\left.\times z_1(2\sqrt{xy})^{-h_1},\ldots,z_r(2\sqrt{xy})^{-h_r}\right)
\end{aligned}\tag{12}$$

Proof. The Laplace transform of the product of a general class of polynomials, Fox's H-function and the multivariable H-function is given by

$$\begin{aligned}
&L\left\{t^{\sigma-1}S^{\mu}_{\lambda}[t^{\delta}]H^{m,n}_{u,d}\left[zt^{k}\middle|\begin{matrix}(e_u,E_u)\\(f_d,F_d)\end{matrix}\right]H^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,w+1:(P',Q');\ldots;(P^{(r)},Q^{(r)})}\left(\begin{matrix}[(a):A',\ldots,A^{(r)}]:\\ [(c):C',\ldots,C^{(r)}],\end{matrix}\right.\right.\\
&\times[1-\sigma-kg_s-\delta\alpha:h_1,\ldots,h_r]:\begin{matrix}[(b'):B'];\ldots;[(b^{(r)}):B^{(r)}];\\ [(d'):D'];\ldots;[(d^{(r)}):D^{(r)}];\end{matrix}\\
&\left.\left.\times z_1t^{h_1},\ldots,z_rt^{h_r}\right)\right\}\\
&=p^{-\sigma}\sum_{\alpha=0}^{[\lambda/\mu]}\sum_{G=1}^{m}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!F_G}\phi(g_s)z^{g_s}\frac{(-\lambda)_{\mu\alpha}}{\alpha!}L_{\lambda,\alpha}\cdot p^{-kg_s-\delta\alpha}\\
&\times H^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}(z_1p^{-h_1},\ldots,z_rp^{-h_r}),
\end{aligned}\tag{13}$$

where

$$\mathrm{Re}(p) > 0, \quad \mathrm{Re}\left(\sigma + k\theta + \sum_{i=1}^{r} h_i\theta_i\right) > 0,$$

$$\mathrm{Re}\left(\sigma + k\phi + \sum_{i=1}^{r} h_i\phi_i\right) < 0, \quad h_i > 0, \quad k > 0, \quad \delta > 0,$$

$$|\arg(z_i)| < T_i\pi/2, \quad |\arg(z)| < T\pi/2, \quad T_i > 0, \quad T > 0,$$

μ is an arbitrary positive integer and the coefficients $L_{\lambda,\alpha}$ $(\lambda,\alpha \geqslant 0)$ are arbitrary constants real or complex. The result in (13) can easily be established by making use of (3) and a result recently obtained by Chaurasia [3, eqn. (2.2), p. 22].

On writing $(pq)^{-1/2}$ for p and multiplying both sides of (13) by $p^{-1/2}(pq)^{1-\rho}$ and then interpreting it with the help of a known result [4, p. 144, eqn. (3.26)], we get

$$\frac{(4xy)^{(\rho/2)-(1/4)}}{\sqrt{\pi y}} \int_0^\infty t^{\sigma-\rho-(1/2)} J_{2\rho-1}[(64xyt^2)^{1/4}]$$

$$\times H_{u,d}^{m,n}\left[zt^k \begin{matrix} (e_u, E_u) \\ (f_d, F_d) \end{matrix} \right] \cdot S_\lambda^\mu[t^\delta] H_{v,w+1:(P',Q');\ldots;(P^{(r)},Q^{(r)})}^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}$$

$$\times \left(\begin{matrix} [(a):A',\ldots,A^{(r)}]: \\ [(c):C',\ldots,C^{(r)}], \end{matrix} [1-\sigma-kg_s-\delta\alpha:h_1,\ldots,h_r]: \right.$$

$$\left. \begin{matrix} [(b'):B'];\ldots;[(b^{(r)}:B^{(r)}]; \\ [(d'):D'];\ldots;[(d^{(r)}):D^{(r)}]; \end{matrix} z_1 t^{h_1},\ldots,z_r t^{h_r} \right) \mathrm{d}t$$

$$\doteqdot p^{-1/2}(pq)^{(\sigma/2)-\rho+1} \sum_{\alpha=0}^{[\lambda/\mu]} \sum_{G=1}^{m} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!F_G} \phi(g_s) z^{g_s}$$

$$\times \frac{(-\lambda)_{\mu\alpha}}{\alpha!} L_{\lambda,\alpha} (pq)^{(kg_s+\delta\alpha)/2} \cdot H_{v,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}$$

$$\times (z_1(\sqrt{pq})^{h_1},\ldots,z_r(\sqrt{pq})^{h_r}). \tag{14}$$

Now, evaluating LHS of (14) by the process mentioned in (13) we obtain the described result and (21) is established.

2. Special cases

Letting $K \to 0$, from (12) we get (after a little simplification) the following bivariate Laplace transform for a general class of polynomials and the multivariable H-function in the elegant form.

Theorem 2. *With T_i, θ_i and ϕ_i given by (7) and (8) respectively, let $T_i > 0$, $|\arg(z_i)| < T_i\pi/2$, $h_i > 0$, $i = 1,\ldots,r$, $\delta > 0$, μ is an arbitrary positive integer and the coefficients $L_{\lambda,\alpha}(\lambda,\alpha \geqslant 0)$ are arbitrary constants real or complex, and*

(i) $\mathrm{Re}\left(\sigma + \sum_{i=1}^{r} h_i\theta_i\right) > 0$

(ii) $\mathrm{Re}\left(\rho - \sigma - \sum_{i=1}^{r} h_i\phi_i\right) < 3/4.$

Also let
(iii) $\mathrm{Re}(p) > 0, \quad \mathrm{Re}(\rho) > 0$

$$p^{-1/2}(pq)^{(\sigma/2)-\rho+1}\sum_{\alpha=0}^{[\lambda/\mu]}\frac{(-\lambda)_{\mu\alpha}}{\alpha!}L_{\lambda,\alpha}(pq)^{\delta\alpha/2}$$
$$\times H^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}(z_1(\sqrt{pq})^{h_1},\ldots,z_r(\sqrt{pq})^{h_r})$$
$$\doteqdot \frac{(4xy)^{\rho-(\sigma/2)-(1/2)}}{\sqrt{\pi y}}\sum_{\alpha=0}^{[\lambda/\mu]}\frac{(-\lambda)_{\mu\alpha}}{\alpha!}L_{\lambda,\alpha}(4xy)^{-\delta\alpha/2}.$$
$$\times H^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v+1,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}\left(\begin{matrix}[(a):A',\ldots,A^{(r)}],\\ [(c):C',\ldots,C^{(r)}]:\end{matrix}[2\rho-\sigma-\delta\alpha:h_1,\ldots,h_r]:\right.$$
$$\left.\times\begin{matrix}[(b'):B'];\ldots;[(b^{(r)}):B^{(r)}];\\ [(d'):D'];\ldots;[(d^{(r)}):D^{(r)}];\end{matrix}z_1(2\sqrt{xy})^{-h_1},\ldots,z_r(2\sqrt{xy})^{-h_r}\right) \tag{15}$$

(ii) *Letting* $\lambda\to 0$, *the theorem* 1 *reduces to a known theorem recently obtained by Chaurasia* [3, *eqn.* (2.1), *p.* 21].
(iii) *Letting* $\lambda\to 0$, $K\to 0$ *and* $r=2$, *the theorem* 1 *reduces to a known result* [2, *eqn.* (2.1), *p.* 86].
(iv) *Putting* $\lambda=0$ *in* (15), *we get a known result recently obtained by Chaurasia* [3, *eqn.* (3.1), *p.* 24].
(v) *Taking* $r=1$ *and* $\lambda\to 0$, *the theorem 2 reduces to a known theorem obtained in* [2, *theorem* (3b), *p.* 88].

The importance of our results lies in its manifold generality. In view of the generality of the polynomials $s^{\mu}_{\lambda}[x]$, on suitably specializing the coefficients $L_{\lambda,\alpha}$, and making a free use of the special cases of $S^{\mu}_{\lambda}[x]$ listed by Srivastava and Singh [9], our results can be reduced to a large number of bivariate Laplace transforms involving generalized Hermite polynomials, Hermite polynomials, Jacobi polynomials and its various special cases, Laguerre polynomials, Bessel polynomials, Gould-Hopper polynomials, Brafman polynomials and their various combinations.

Secondly, by specializing the various parameters and variables in Fox's H-function and in the multivariable H-function, from our results, several bivariate Laplace transforms involving a remarkably wide variety of useful functions (or products of several such functions), which are expressible in terms of E, F, G and H functions of one and several variables. Thus the results presented in this paper would at once yield a very large number of results, involving a large variety of polynomials and various special functions occurring in the problems of mathematical analysis.

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Some applications of Briot–Bouquet differential subordination

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Abstract. Some applications of Briot–Bouquet differential subordination are obtained which improve or extend a number of classical results in the univalent function theory.

Keywords. Analytic functions; Ruscheweyh derivatives; differential subordination.

1. Introduction

Let V denote the class of functions f of the form $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disc $E = \{z : |z| < 1\}$. For given arbitrary numbers A and B satisfying $-1 \leqslant B < A \leqslant 1$, we denote by $P(A, B)$, the class of functions of the form $p(z) = 1 + p_1 z + \dots$ which are analytic in E and satisfy the condition

$$p(z) \ll \frac{1 + Az}{1 + Bz}, \quad z \in E,$$

where "$\ll$" stands for subordination. The class $P(A, B)$ was investigated by Janowski [4].

By $S^*(A, B)$, we mean the class of functions $f \in V$ such that $zf'(z)/f(z) \in P(A, B)$. Similarly, $K(A, B)$ is meant as the class of functions $f \in V$ satisfying $(zf'(z))'/f'(z) \in P(A, B)$. It is clear that $S^*(1 - 2\rho, -1) \equiv S^*(\rho)$ and $K(1 - 2\rho, -1) \equiv K(\rho)$, $0 \leqslant \rho < 1$, $S^*(\rho)$ and $K(\rho)$ denote the subclasses of V which are respectively star-like of order ρ and convex of order ρ in E.

We now define the class $KS^*_{\delta,\alpha}(A, B)$ as below and obtain some interesting results by an application of Briot–Bouquet differential subordination.

DEFINITION 1

Let A, B, α and δ be arbitrary fixed real numbers such that $-1 \leqslant B < A \leqslant 1$, $\alpha \geqslant 0$ and $\delta \geqslant -1$. A function $f \in V$ is said to be in the class $KS^*_{\delta,\alpha}(A, B)$ if it satisfies

$$J_\delta(\alpha; f(z)) \ll \frac{1 + Az}{1 + Bz}, \quad z \in E, \tag{1}$$

where

$$J_\delta(\alpha; f(z)) = (1 - \alpha)\frac{z(D^\delta f(z))'}{D^\delta f(z)} + \alpha \frac{D^{\delta+2} f(z)}{D^{\delta+1} f(z)}$$

and

$$D^{\delta}f(z)=(z/(1-z)^{\delta+1})*f(z)$$

(Here $*$ stands for the Hadamard product of two analytic functions).

Clearly $KS^*_{0,0}(A,B)\equiv S^*(A,B)$, $KS^*_{0,1}((A+B)/2,B)\equiv K(A,B)$. It is readily seen that $KS^*_{n,\alpha}(1-2\rho,-1)$, $\rho<1$, $n\in N_0=\{0,1,2,\ldots\}$ is the class defined by Bulboaca [3], whereas the class

$$KS^*_{n,0}\left(\frac{b^2-a^2+a}{b},\frac{1-a}{b}\right),\quad |a-1|<b\leqslant a,$$

is the class considered by us [2]. Further taking $\delta=0$, $\alpha=2\mu/(\mu+1)(\mu\geqslant 0)$, $A=1-2\mu/(\mu+1)$, $B=-1$, it is seen that the class $KS^*_{\delta,\alpha}(A,B)$ reduces to the well-known class of μ-convex functions ([5] and [6]).

For a,b,c real numbers with c other than $0,-1,-2,\ldots$ the hypergeometric series

$$F(a,b,c;z)=1+\frac{a\cdot b}{1\cdot c}z+\frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}z^2+\ldots$$

represents an analytic function in E[11]. The following identities are well known [11, Chapter 15].

$$F(a,b,c;z)=F(b,a,c;z) \tag{2}$$

$$F(a,b,c;z)=(1-z)^{-a}F(a,c-b,c;z/(z-1)),\quad z\in C-\{1,\infty\} \tag{3}$$

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\mathrm{d}t=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F(a,b,c;z) \tag{4}$$

with $c>b>0$.

2. Main results

Theorem 2.1 *Let* $-1\leqslant B<A\leqslant 1$, $\delta>-1$ *and* $0<\alpha<(\delta+2)/(\delta+1)$ *satisfy*

$$(\delta+2)(1-A)+[\delta(\delta+2)(1-\alpha)-\alpha](1-B)\geqslant 0. \tag{5}$$

(a) *Then*

$$KS^*_{\delta,\alpha}(A,B)\subset KS^*_{\delta,0}(A',B) \tag{6}$$

where

$$A'=1-\frac{(\delta+2)(1-A)-\alpha(\delta+1)(1-B)}{(\delta+2)-\alpha(\delta+1)}.$$

Further for $f\in KS^*_{\delta,\alpha}(A,B)$ *we also have*

$$\frac{z(D^{\delta}f(z))'}{D^{\delta}f(z)}\ll\frac{\alpha}{[(\delta+2)-\alpha(\delta+1)]}\left\{\frac{1}{Q(z)}\right\}-\delta\equiv\tilde{q}(z),\quad z\in E \tag{7}$$

where

$$Q(z)=\begin{cases}\displaystyle\int_0^1\left(\frac{1+Btz}{1+Bz}\right)^{((\delta+2)/\alpha)((A-B)/B)}t^{((\delta+2)/\alpha)(\delta(1-\alpha)+1)-2}\mathrm{d}t & \text{if } B\neq 0\\ \displaystyle\int_0^1\exp\left\{\frac{\delta+2}{\alpha}(t-1)Az\right\}t^{((\delta+2)/\alpha)(\delta(1-\alpha)+1)-2}\mathrm{d}t & \text{if } B=0.\end{cases} \tag{8}$$

(b) *If in addition to* (5) *one has* $A/B > -\delta(1-\alpha)$ *with* $B<0$, *then*

$$KS^*_{\delta,\alpha}(A,B) \subset KS^*_{\delta,0}(1-2\rho', -1) \tag{9}$$

where

$$\rho' = \frac{\delta+1}{F\left(1, \left(\frac{\delta+2}{\alpha}\right)\left(\frac{B-A}{B}\right), \left(\frac{\delta+2}{\alpha}\right)(\delta(1-\alpha)+1); \frac{-B}{1-B}\right)} - \delta.$$

The result is sharp.

Proof. (a) We follow the method similar to that of Mocanu *et al* [8]. It can be easily verified that

$$z(D^\delta f(z))' = (\delta+1)D^{\delta+1}f(z) - \delta D^\delta f(z). \tag{10}$$

Let $f \in KS^*_{\delta,\alpha}(A,B)$ where $\delta > -1$, $\alpha > 0$ and $-1 \leqslant B < A \leqslant 1$. Set

$$\tilde{p}(z) = \frac{z(D^\delta f(z))'}{D^\delta f(z)}. \tag{11}$$

Clearly $\tilde{p}(z)$ is analytic in E and $\tilde{p}(o) = 1$. From (10) and (11) we obtain

$$\frac{D^{\delta+1}f(z)}{D^\delta f(z)} = \frac{\tilde{p}(z)+\delta}{1+\delta} = p(z); \quad p(o) = 1.$$

Since $f \in KS^*_{\delta,\alpha}(A,B)$, (1) coupled with (10) easily leads to

$$P(z) + \frac{zP'(z)}{\beta P(z)+\gamma} \ll \frac{1+Az}{1+Bz}, \quad z\in E \tag{12}$$

where

$$P(z) = (\delta+1)\left(\frac{1}{\beta}+1-\alpha\right)p(z) + \frac{1}{\beta} - \delta(1-\alpha) \tag{13}$$

with $\beta = (\delta+2)/\alpha$ and $\gamma = \delta\beta(1-\alpha) - 1$. Considering the differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z)+\gamma} = \frac{1+Az}{1+Bz}, \quad z\in E$$

and using Corollary 3.2 of Miller and Mocanu [7] we deduce that

$$P(z) \ll q(z) \ll \frac{1+Az}{1+Bz}, \quad z\in E \tag{14}$$

provided δ, α, A and B satisfy (5), where $q(z)$ is the best dominant of (12) and is given by (23) in [7, Corollary 3.2]. (6) and (7) now follow from (13) and (14).

(b) Next we show that

$$\operatorname*{Inf}_{z\in E}\{\tilde{q}(z)\} = \tilde{q}(-1). \tag{15}$$

If we set $a = \beta(B-A)/B$, $b = \beta+\gamma$, $c = \beta+\gamma+1$, then $c>b>0$. From (8) by using

(2), (3) and (4) we see that for $B \neq 0$

$$\begin{aligned} Q(z) &= (1+Bz)^a \int_0^1 (1+Btz)^{-a} t^{b-1}\,\mathrm{d}t \\ &= (1+Bz)^a \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}(1+Bz)^{-a} F\left(a, c-b, c; \frac{Bz}{Bz+1}\right) \\ &= \frac{1}{b} F\left(1, a, c; \frac{Bz}{Bz+1}\right). \end{aligned}$$

Supposing $-1 \leqslant B < A < 0$ we have $c > a > 0$ and by using (4) we obtain

$$Q(z) = \int_0^1 \left[\frac{1+Bz}{1+(1-t)Bz}\right] \mathrm{d}\mu(t)$$

where

$$\mathrm{d}\mu(t) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)} t^{a-1}(1-t)^{c-a-1}\,\mathrm{d}t$$

is a positive measure. Denoting $g(z,t) = (1+Bz)/(1+(1-t)Bz)$, it may be seen that $\operatorname{Re}\{g(z,t)\} > 0$, $g(-r,t)$ is real for $0 \leqslant r < 1$, $t \in [0,1]$ and

$$\operatorname{Re}\left\{\frac{1}{g(z,t)}\right\} = \operatorname{Re}\left\{\frac{1+(1-t)Bz}{1+Bz}\right\} \geqslant \frac{1-(1-t)Br}{1-Br} = \frac{1}{g(-r,t)}$$

for $|z| \leqslant r < 1$ and $t \in [0,1]$. Therefore by using Lemma 2 [12] we deduce that $\operatorname{Re}(1/Q(z)) \geqslant 1/(Q(-r)), |z| \leqslant r < 1$ and letting $r \to 1^-$ we obtain $\operatorname{Re}(1/Q(z)) \geqslant 1/(Q(-1))$, $z \in E$. This by (15) and (7) leads to (9). Hence the theorem.

We write $KS^*_{\delta,0}(A,B) \equiv S^*_\delta(A,B)$ and $KS^*_{\delta,1}(A,B) = K_{\delta+1}(A,B)$. Then it can be readily seen, using the identity (10), that

$$S^*_\delta(A,B) \equiv K_\delta\left(\frac{A+\delta B}{1+\delta}, B\right) \text{ or } K_\delta(A,B) \equiv S^*_\delta((\delta+1)A - \delta B, B) \tag{16}$$

provided $\delta > 0$. Further we have $S^*_\delta(1-2\rho, -1) \equiv S^*_\delta(\rho)$ and $K_\delta(1-2\rho, -1) \equiv K_\delta(\rho)$ for $\rho < 1$.

Taking $\alpha = 1$ in Theorem 2.1 (a) and using (16) we obtain the following result of Ramareddy [9].

COROLLARY 2.1

Let $-1 \leqslant B < A \leqslant 1$ *and* $\delta > -(1-A)/(1-B)$. *Then* $S^*_{\delta+1}(A,B) \subset S^*_\delta(A,B)$.

Taking $A = 1-2\rho$ and $B = -1$ in Theorem 2.1 we get

COROLLARY 2.2

Let $\delta > -1$, $0 < \alpha < (\delta+2)/(\delta+1)$ *and* $\dfrac{\alpha - \delta(\delta+2)(1-\alpha)}{\delta+2} \leqslant \rho < 1$. *Then*

$$KS^*_{\delta,\alpha}(1-2\rho, -1) \subset S^*_\delta\left(1 - 2\frac{(\delta+2)\rho - \alpha(\delta+1)}{(\delta+2) - \alpha(\delta+1)}, -1\right). \tag{17}$$

Further if $f \in KS^*_{\delta.\alpha}(1-2\rho,-1)$, then

$$\frac{z(D^\delta f(z))'}{D^\delta f(z)} \ll \frac{\alpha}{[(\delta+2)-\alpha(\delta+1)]}\left[\int_0^1 \left(\frac{1-tz}{1-z}\right)^{[-2(\delta+2)(1-\rho)]/\alpha} \times t^{(\delta+2)/\alpha)(\delta(1-\alpha)+1)-2}\,\mathrm{d}t\right]^{-1} - \delta.$$

Furthermore if $\max\left\{\frac{\alpha-\delta(\delta+2)(1-\alpha)}{\delta+2}, \frac{1-\delta(1-\alpha)}{2}\right\} \leqslant \rho < 1$,

then

$$KS^*_{\delta,\alpha}(1-2\rho,-1) \subset S^*_\delta(1-2\rho'',-1) \tag{18}$$

where

$$\rho'' = (\delta+1)\left[F\left\{1, \frac{2(\delta+2)(1-\rho)}{\alpha}, \frac{(\delta+2)}{\alpha}(\delta(1-\alpha)+1); \frac{1}{2}\right\}\right]^{-1} - \delta.$$

The result is sharp.

Remarks i) For $\delta = n \in N_0$, (17) and (18) were obtained by Bulboaca [3].

ii) If we take $\alpha = 1$, $\rho = (\delta+1)/(\delta+2)$ in Corollary 2.2, it follows, from (18), that for $f \in V$ and $\delta \geqslant 0$ we have the sharp result $\mathrm{Re}\{D^{\delta+2}f(z)/D^{\delta+1}f(z)\} > (\delta+1)/(\delta+2)$ in E implies $\mathrm{Re}\{z(D^\delta f(z))'/D^\delta f(z)\} > ((\delta+1)/F(1,2,\delta+2;1/2)) - \delta$ in E which in turn implies $\mathrm{Re}\{D^{\delta+1}f(z)/D^\delta f(z)\} > 1/F(1,2,\delta+2;1/2) > \delta/(\delta+1)$, $z \in E$. This improves Singh and Singh's results [10] obtained for $\delta = n \in N_0$.

(iii) Letting $\alpha = 1$ in Corollary 2.2, we obtain that, for

$$\rho \in \left(\max\left\{\frac{1}{\delta+2}, \frac{1}{2}\right\}, 1\right),\ K_{\delta+1}(\rho) \subset S^*_\delta(\rho''),$$

where ρ'' is as stated in Corollary 2.2 with $\alpha = 1$. Using $K_\delta(\rho) = S^*_\delta((\delta+1)\rho - \delta)$, this result takes the form $S^*_{\delta+1}(\rho) \subset S^*_\delta(\rho^*)$ for $-(\delta/2) \leqslant \rho < 1$, where $\rho^* = (\delta+1)[F(1,2(1-\rho),\delta+2;1/2]^{-1} - \delta$, which represents an improvement and extension of Ahuja's result [1] obtained for $\delta = n \in N_0$ and $0 \leqslant \rho < 1$.

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Construction techniques for Galois coverings of the affine line

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Abstract. For constructing unramified coverings of the affine line in characteristic p, a general theorem about good reductions modulo p of coverings of characteristic zero curves is proved. This is applied to modular curves to realize $\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})/\{\pm 1\}$, with $\mathrm{GCD}(n, 6) = 1$, as Galois groups of unramified coverings of the affine line in characteristic p, for $p = 2$ or 3. It is applied to the Klein curve to realize $\mathrm{PSL}(2, 7)$ for $p = 2$ or 3, and to the Macbeath curve to realize $\mathrm{PSL}(2, 8)$ for $p = 3$. By looking at curves with big automorphism groups, the projective special unitary groups $\mathrm{PSU}(3, p^\nu)$ and the projective special linear groups $\mathrm{PSL}(2, p^\nu)$ are realized for all p, and the Suzuki groups $\mathrm{Sz}(2^{2\nu+1})$ are realized for $p = 2$. Jacobian varieties are used to realize certain extensions of realizable groups with abelian kernels.

Keywords. Good reduction; affine line; unramified covering; Galois group.

1. Introduction

In characteristic zero, the affine line is simply connected and therefore has no nontrivial unramified covering. In characteristic $p > 0$, however, only the prime-to-p part of its algebraic fundamental group is isomorphic to the prime-to-p part of the algebraic fundamental group of the corresponding characteristic zero curve and hence trivial. Its p-part, however, is nontrivial; in fact by a conjecture of Abhyankar [A03] it should be as nontrivial as can be: *Every group which is generated by its p-Sylow subgroups should occur as a Galois group of an unramified covering of the affine line.* Such groups are called *quasi p-groups.*

In the last few years, in support of this conjecture, several quasi p-groups have been realized as Galois groups of unramified coverings of the affine line in characteristic p; see [A10] to [A16], [AOS], [APS], [AYi], [Nor], and [Se2]; for a summary see [A14] to [A16]. It may be noted that the above *quasi p-group conjecture* is a special case of Abhyankar's *general conjecture* [A03] according to which the algebraic fundamental group $\pi_A(C_{g,w})$ coincides with the set of all finite groups G for which $G/p(G)$ is generated by $2g + w$ generators where $p(G)$ denotes the subgroup of G generated by all of its p-Sylow subgroups. Here $C_{g,w} = C_g \backslash \{w + 1 \text{ points}\}$ where w is a nonnegative integer and C_g is a nonsingular projective curve of genus g over an algebraically closed field of characteristic p. Recall that $\pi_A(C_{g,w})$ is defined to be the set of all finite groups which can be realized as Galois groups of unramified coverings of $C_{g,w}$. Thus, upon letting $\mathscr{A}_k^1$ = the affine line over an algebraically closed field k of characteristic p, and $Q(p)$ = the set of all quasi p-groups, the quasi p-group conjecture predicts that: $\pi_A(\mathscr{A}_k^1) = Q(p)$.

One aim of this paper is to construct unramified coverings of the affine line for some more quasi p-groups, mostly for simple groups or for groups which are closely related to simple groups. Note that a simple group is a quasi p-group iff its order is divisible by p, because then the p-Sylow subgroups generate a nonidentity normal subgroup.

A possible technique for doing this is the following: Having realized a group G of the type we are interested in as an automorphism group of an algebraic curve $\mathscr{X}$ in characteristic zero, we consider a reduction $\mathscr{X}'$ of $\mathscr{X}$ modulo p. Briefly speaking, the curve $\mathscr{X}$ can be given by various sets of polynomial equations with integer coefficients, and the curve $\mathscr{X}'$ over the finite field GF(p) is obtained by reducing the coefficients of a suitable set of equations of $\mathscr{X}$ modulo p. To compare the Galois theories of $\mathscr{X}$ and $\mathscr{X}'$ we regard them as the generic and special fibers E_T and $\mathfrak{Z}(T_E)$ of an arithmetic surface E fibered over a discrete valuation ring T. At any rate, if $\mathscr{X}'$ is irreducible and the quotient $\mathscr{X}'/G$ is the projective line then the quotient map $\mathscr{X}' \to \mathscr{X}'/G$ is a first candidate for the type of covering we are looking for, and we try to remove unwanted ramification points using Abhyankar's Lemma. In § 2 we prove a general theorem (2.4) of this type, which (supplemented by (2.6)) is a principal result of this paper, and which briefly says that the prime-to-p parts of the ramification exponents of those point of the generic fiber which specialize to a given point of the special fiber are pairwise coprime and their product is the prime-to-p part of the ramification exponent of the given point of the special fiber, whereas the p-parts of the ramification exponents can (and often will) increase. In § 1 we prepare the ground work for this; in particular, Proposition (1.4) is a local version of Theorem (2.4). In Remark (2.7) of § 2 we give necessary conditions for a characteristic zero curve $\mathscr{X}$ to have a good reduction modulo p, i.e., briefly speaking, for the existence of equations for $\mathscr{X}$ which when reduced modulo p give a nonsingular irreducible curve.

In § 3 we apply the general theorem to modular curves to get $\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})/\{\pm 1\}$ with $\mathrm{GCD}(n, 6) = 1$ for $p = 2$ or 3, where by "get" we mean "realize as Galois groups of unramified coverings of the affine line"; see (3.2). Likewise, in § 4 we apply the general theorem to the Klein curve and the Macbeath curve to get PSL(2, 7) for $p = 2$ or 3 and PSL(2, 8) for $p = 3$; see (4.1.1) and (4.1.2); the Macbeath case was also done by Serre [Se5] and [Se6]. Note that the Klein curve and the Macbeath curve are curves of genus $g = 3$ and $g = 7$ in characteristic zero with automorphism groups of order $84(g - 1)$ which is maximum possible by Hurwitz's Theorem. In § 4 we also discuss the fact that, as discovered by Leopoldt [Leo], Stichtenoth [Sti] and Henn [Hen], automorphism groups of curves in positive characteristic can be much bigger than in characteristic zero, and by directly applying Abhyankar's Lemma to some of these curves with big automorphism groups we get the projective special unitary group $\mathrm{PSU}(3, p^\nu)$ and the projective special linear group $\mathrm{PSL}(2, p^\nu)$ for every p and the Suzuki group $\mathrm{Sz}(2^{2\nu+1})$ for $p = 2$, where ν is any positive integer; see (4.2.1) to (4.2.3); other methods of getting $\mathrm{PSL}(2, p^\nu)$ can be found in [A10] and [A16], and other methods of getting $\mathrm{PSU}(3, p^\nu)$ and $\mathrm{Sz}(2^{2\nu+1})$ were indicated by Serre in [Se3]. In (5.3) of § 5, by using Jacobian varieties we show that given any $G \in \pi_A(L_k)$, for every positive integer $l \not\equiv 0(p)$ and for infinitely many positive integers γ, some extension of G with kernel $(\mathbb{Z}/l\mathbb{Z})^{2\gamma}$ belongs to $\pi_A(L_k)$; in some sense, this may be regarded as a special case of Serre's recent result [Se2] according to which every quasi-p extension of G with solvable kernel belongs to $\pi_A(\mathscr{A}_k^1)$; the special case is also indicated in Serre [Se4]. Finally, in § 6 we pose some problems.

1. Some lemmas from local algebra

Geometric outline: Let T be a discrete valuation ring with quotient field k^*, and let k' be the residue field of T, i.e., $k' = T/M(T)$ where $M(T)$ is the maximal ideal in T. Usually, the characteristic of k^* will be zero and the characteristic of k' will be $p > 0$; for instance T could be the localization of $\mathbb{Z}$ at prime p. Let $\mathscr{X} \to \mathscr{X}_0$ be a covering of curves over k^* and let $\mathscr{X}' \to \mathscr{X}'_0$ be the covering of curves over k' obtained by reducing it modulo p; we shall be mostly interested in the case when $\mathscr{X}_0$ and $\mathscr{X}'_0$ are the projective lines $\mathscr{P}_{k^*}^1$ and $\mathscr{P}_{k'}^1$ over k^* and k' respectively. To compare the Galois theories of these two coverings, we regard $\mathscr{X}$ and $\mathscr{X}'$ as the generic and special fibers E_T and $\mathfrak{Z}(T_E)$ of an arithmetic surface E over T, (where $\mathfrak{Z}(T_E)$ is the zero-set of the ideal T_E which will be defined in § 2). To visualize the surface E geometrically, as a first shot, we may think of it as consisting of two nonintersecting vertical curves, the generic fiber E_T on the left, and the special fiber $\mathfrak{Z}(T_E)$ on the right. Now it is not easy to compare the two curves E_T and $\mathfrak{Z}(T_E)$ when they are sitting isolatedly without any intermediaries between them. So we replace every point of the generic fiber E_T by a horizontal curve meeting the special fiber $\mathfrak{Z}(T_E)$ in the corresponding point. But remember that many points on the characteristic zero curve have the same reduction modulo p; for example the points $x = 1$, $p + 1$, $2p + 1, \ldots$ are all reduced to the point $x = 1$. Thus, whereas a point of E_T uniquely determines its reduction on $\mathfrak{Z}(T_E)$, many horizontal curves are going to meet at a common point Q on the special fiber; indeed infinitely many! So the correspondence between E_T and $\mathfrak{Z}(T_E)$ is infinite to one, and that is why it is difficult to draw a good geometric picture.* At any rate, $\mathscr{X}_0$ and $\mathscr{X}'_0$ may also be regarded as the generic and special fibers of an arithmetic surface D over T, giving us a covering of surfaces $E \to D$. Postponing the global comparison of the Galois theories of $\mathscr{X}$ and $\mathscr{X}'$ to § 2, here we study the matter locally. To do this, choose a point P of $\mathfrak{Z}(T_D)$, let R be the local ring of P as a point of D, and let B be the semilocal ring of its inverse image on E. Also let Θ and U be the localizations of B and R at the prime ideals in them generated by $M(T)$. Note that then Θ and U are the local rings of $\mathfrak{Z}(T_E)$ and $\mathfrak{Z}(T_D)$ on E and D respectively. In any case Θ is a discrete valuation ring whose quotient field L coincides with the function field of $\mathscr{X}$, i.e., with the function field of E, and likewise U is a discrete valuation ring whose quotient field K coincides with the function field of $\mathscr{X}_0$, i.e., with the function field of D. Note that if Q lies above P then the local ring S of Q on E is the localization of B at a maximal ideal, and Θ together with the local rings (on E) of the infinitely many horizontal curves meeting in Q are exactly all the localizations of S at its nonzero minimal prime ideals and, by one of Krull's many theorems, S (being normal) coincides with their intersection; so the geometry of infinitely many curves meeting in a point is reproduced algebraically. Finally let K' and L' be the residue fields of U and Θ

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*The other difficulty, namely the fact that only one curve passes through a point of the generic fiber, can be mollified by thinking of the generic fiber as a very fat curve whose points are so fat that, like the germs of analytic functions, each of them uniquely determines a curve passing through it.

respectively. Then by reducing R and B modulo p, i.e., by taking their images under the maps $U \to K'$ and $\Theta \to L'$, we get the local ring R' of P on $\mathscr{X}'_0$ and the semilocal ring B' of its inverse image on $\mathscr{X}'$. All this may be schematically depicted in the following diagrams, where we note that the second column in the right-hand diagram starting with Θ does not correspond to the second column in the left-hand diagram starting with E_T but it corresponds to the fourth column in the left-hand diagram starting with $\mathfrak{Z}(T_E)$.

$$\begin{tikzcd}
\mathscr{X} \arrow[r, phantom, "="] \arrow[d] & E_T \arrow[r] \arrow[d] & E & \mathfrak{Z}(T_E) \arrow[l] \arrow[r, phantom, "\approx"] \arrow[d] & \mathscr{X}' \arrow[d] \\
\mathscr{X}_0 \arrow[r, phantom, "="] \arrow[d, dash] & D_T \arrow[r] \arrow[d, dash] & D \arrow[d, dash] & \mathfrak{Z}(T_1) \arrow[l] \arrow[r, phantom, "\approx"] \arrow[d, dash] & \mathscr{X}'_0 \arrow[d, dash] \\
k^* \arrow[r, phantom, "="] & k^* & T \arrow[l] \arrow[r] & k' \arrow[r, phantom, "="] & k'
\end{tikzcd}
\qquad
\begin{tikzcd}
L & \Theta \arrow[l] & B \arrow[l] \arrow[r] & B' \arrow[r] & L' \\
K \arrow[u] & U \arrow[l] \arrow[u] & R \arrow[l] \arrow[r] \arrow[u] & R' \arrow[r] \arrow[u] & K' \arrow[u] \\
k^* \arrow[u] & T \arrow[l] \arrow[r, phantom, "="] \arrow[u] & T \arrow[r] \arrow[u] & k' \arrow[r, phantom, "="] \arrow[u] & k' \arrow[u]
\end{tikzcd}$$

Actually, in this section we are only concerned with the right-hand diagram. To recapitulate, B and B' are corresponding "semilocal pieces" of an arithmetic surface E and its special fiber $\mathfrak{Z}(T_E)$ whose local ring is Θ, and we want to relate the Galois theory of the points of B with the Galois theory of the points of B'; we do this by interposing the Galois theory of the above mentioned "horizontal curves" passing through the points of B; these horizontal curves correspond to those points of the generic fiber E_T of E whose specializations are the points of B', i.e., which when reduced modulo p give the points of B'. At any rate, we start with three preparatory Lemmas (1.1) to (1.3), and then we prove the main result of this section in Proposition (1.4), which, briefly, says that the prime-to-p parts of the ramification exponents of those point of the generic fiber which specialize to a given point of the special fiber are pairwise coprime and their product is the prime-to-p part of the ramification exponent of the given point of the special fiber, whereas the p-parts of the ramification exponents can (and often will) increase. In Lemmas (1.1) and (1.2) we deal with certain localized versions of Proposition (1.4); these localized versions remain valid in higher dimensions. In Lemma (1.3) we give a sufficient condition for two curves on a surface to meet transversally. All this is put together in Proposition (1.4). Then in Proposition (1.5) we recall some properties of prime divisors of second kind, and in Corollary (1.6) we deduce some of their consequences. In particular we shall see that $D_T \approx \mathscr{P}^1_{k^*}$ implies $\mathfrak{Z}(T_D) \approx \mathscr{P}^1_{k'}$ and that both the lines have a common "coordinate". First let us fix some algebraic terminology.

Algebraic terminology: We shall use the terminology of quadratic transformations as given in [A01], the terminology of Galois theory as given in Chapter 1 of [A04], and the terminology of models as given in §2 and §3 of [A07] and §1 and §6 of [A08]. In particular, $\mathfrak{B}(A)$ stands for the set of all localizations of a domain A with respect to the various prime ideals in A. The maximal ideal in a local ring R is denoted by $M(R)$. A ring A (always commutative with 1) is said to be *pseudogeometric* if A is noetherian and, for every domain which is a homomorphic image of A, we have that its integral closure in any finite algebraic field extension of its quotient field is a finite module over it. A noetherian domain A is said to be *regular* if all members of $\mathfrak{B}(A)$ are regular. By a DVR we mean a discrete valuation ring. For $V \in \mathfrak{B}(R)$ where R is a local domain, *V has a simple point at R* means the local domain $R/(M(V) \cap R)$ is regular; if V is a DVR and R is regular then this is equivalent to saying that $M(V) \cap R \not\subset M(R)^2$. By a *prime divisor of second kind of a regular local domain R* we mean a DVR which birationally dominates R and for whose residual transcendence degree δ over R we have $\delta \geqslant (\dim R)-1$ [and hence $\delta = (\dim R)-1$]. Note that *birationally dominates* means dominates and has a common quotient field. Given a local domain U and a subring A of U, by a *residual transcendental generator of* U *over* A we mean an element x of U (if it exists) such that upon letting $\omega : U \to U/M(U)$ to be the canonical epimorphism and $\kappa =$ the quotient field of $\omega(A)$ we have that $\omega(x)$ is transcendental over κ and $U/M(U) = \kappa(\omega(x))$.

Given a normal local domain R and a finite algebraic field extension L of the quotient field K of R, by an *extension of* R *to* L we mean a localization of the integral closure of R in L at a maximal ideal; note that then R has at least one and at most a finite number of extensions to L; R is said to be *split* or *unsplit* in L according as it does or does not have more than one extension to L. An extension S of R to L is said to be *naively unramified over* R if $M(R)S = M(S)$; R is said to be *naively unramified in* L if every extension of R to L is naively unramified over R; S is said to be *unramified over* R if it is naively unramified and residually separable algebraic over R; R is said to be *unramified in* L if every extension of R to L is unramified over R; *ramified* means not unramified. If R is a DVR then, for any extension S of R to L, by the *ramification exponent* (resp: *reduced ramification exponent*) *of* S *over* R we mean the ramification exponent (resp: reduced ramification exponent) of ord_S over ord_R, where we note that the ramification exponent divided by the reduced ramification exponent equals the residual degree of inseparability of S over R, i.e., equals 1 or $[S/M(S):R/M(R)]_i$ according as the characteristic of R/M is zero or not; S is said to be *tamely ramified over* R if the ramification exponent of S over R equals its part which is prime to the residue characteristic of R, where by the *residue characteristic* of a local ring we mean the characteristic of its residue field, and for $w=0$ or a prime number, by the *part of a positive integer which is prime to* w we mean the largest factor of that integer which is nondivisible by w; finally, R is said to be *tamely ramified in* L if every extension of R to L is tamely ramified over R. Note that if R is a DVR and L/K is normal then the ramification exponents (resp: reduced ramification exponents, their parts prime to the residue characteristic of R) of the various extensions of R to L coincide with each other. For a finite normal algebraic field extension L/K, as in the Galois (= finite normal separable algebraic) case, by the *Galois group* of L over K we mean the group of all K-automorphisms of L and we denote it by $\operatorname{Gal}(L, K)$.

Note that for any normal noetherian domain A, say because of Krull's Diskriminantensatz (see pages 32 to 34 of [A04]), at most a finite number of DVRs in $\mathfrak{V}(A)$ can be ramified in any given finite separable algebraic field extension of the quotient field of A. This observation, as well as other basic results from the Galois theory of local rings, as given in Chapter 1 of [A04] and Chapter V of [ZSa], may be used tacitly.

Lemma 1.1. *Let* $K \subset L^* \subset L$ *be fields such that* L/L^* *is a finite algebraic extension and* L^*/K *is a cyclic Galois extension whose degree is coprime to the degree of* L/L^*. *Let* d *be a positive integer and let* R *and* S *be* d*-dimensional pseudogeometric regular local domains with quotient fields* K *and* L, *respectively, such that* S *is integral and residually rational over* R. *Assume that* $L^* \neq K$ *and* $[L^*:K]$ *is a power of a prime number which is nondivisible by the characteristic of* $R/M(R)$. *Then* $\mathfrak{V}(R)$ *contains exactly one DVR* V *which is ramified in* L^*. *Moreover* V *is totally ramified in* L^*, *i.e., it has a unique extension to* L^* *and its reduced ramification exponent equals* $[L^*:K]$. *Finally, if* V *has a simple point at* R, *then for every field* K^* *with* $K \subset K^* \subset L^*$ *we have that* $S \cap K^*$ *is a* d*-dimensional regular local domain.*

Proof. Briefly speaking, this follows by extending residue fields, passing to completions, and using standard tricks about Kummer extensions of UFDs as employed in [A02]. In greater detail we may proceed thus.

Let $[L^*:K]=u$. By assumption $u=w^e$ where e is a positive integer, and w is a prime number which is nondivisible by the characteristic of $R/M(R)$. Let ζ_w and ζ_u respectively be primitive wth and uth roots of 1 in an overfield of L. Likewise let ζ'_w and ζ'_u respectively be primitive wth and uth roots of 1 in an overfield of $R/M(R)$. Since L^*/K is cyclic, there is a unique field $L^\#$ with $K\subset L^\#\subset L^*$ such that $L^\#/K$ is cyclic of degree w.

Let $[K(\zeta_w):K]=w^*$. Then $w^*<w$ and $K(\zeta_w)/K$ is an abelian Galois extension. Therefore the fields $L^\#$ and $K(\zeta_w)$ are linearly disjoint over K, and hence $L^\#(\zeta_w)/K(\zeta_w)$ and $L^\#(\zeta_w)/L^\#$ are abelian Galois extensions of degree w and w^* respectively. Let $R^\#_1$, $R^\#_2,\ldots,R^\#_\gamma$ be the distinct extensions of R to $K(\zeta_w)$, and let $[(R/M(R))(\zeta'_w):(R/M(R))]=w'$. Since w is a prime number which is nondivisible by the characteristic of $R/M(R)$, it follows that the Y-discriminant of the polynomial Y^w-1 is a unit in R. Therefore $\gamma w'=w^*$ and for $1\leqslant i\leqslant\gamma$ we have that $R^\#_i$ is unramified over R and $[R^\#_i/M(R^\#_i):R/M(R)]=w'$, and hence in particular $R^\#_i$ is a d-dimensional regular local domain. Let $S^\#=S\cap L^\#$. Then $S^\#$ is the unique extension of R to $L^\#$. Since $S^\#$ is residually rational over R, it follows that $S^\#$ has exactly γ distinct extensions $S^\#_1,S^\#_2,\ldots,S^\#_\gamma$ to $L^\#(\zeta_w)$, and for $1\leqslant i\leqslant\gamma$ we have that $S^\#_i$ is unramified over $S^\#$ and $[S^\#_i/M(S^\#_i):S^\#/M(S^\#)]=w'$. Since $S^\#$ is integral over R, the local rings $S^\#_1,S^\#_2,\ldots,S^\#_\gamma$ can be labelled so that $S^\#_i$ is integral and residually rational over $R^\#_i$ for $1\leqslant i\leqslant\gamma$. Since $L^\#(\zeta_w)/K(\zeta_w)$ is cyclic of degree w, we can write $L^\#(\zeta_w)=K(\zeta_w)(z)$ with $0\neq z^w\in K(\zeta_w)$. Since $R^\#_1$ is a UFD, upon dividing z by a suitable element in $R^\#_1$ we can arrange matters so that $z^w=\varepsilon z_1^{\beta_1}\, z_2^{\beta_2},\cdots,z_\alpha^{\beta_\alpha}$ where β_i is a positive integer nondivisible by w for $1\leqslant i\leqslant\alpha$, the ideals $z_1R^\#_1,z_2R^\#_1\ldots z_\alpha R^\#_1$ are pairwise distinct prime ideals in $R^\#_1$, and ε is a unit in $R^\#_1$. If α were equal to zero then the Y-discriminant of the polynomial Y^w-z^w would be a unit in $R^\#_1$ and hence the sum of the separable parts of the residue degrees of the various extensions of $R^\#_1$ to $L^\#(\zeta_w)$ would equal w, and this would be a contradiction because $S^\#_1$ is the only extension of $R^\#_1$ to $L^\#(\zeta_w)$ and it is residually rational over $R^\#_1$. Therefore we must have $\alpha\neq 0$. Let V be the localization of R at the prime ideal $(z_1R^\#_1)\cap R$.

Now clearly V is a DVR in $\mathfrak{B}(R)$ which is ramified in $L^\#$. Since L^*/K is cyclic of prime power degree nondivisible by the characteristic of $R/M(R)$, it follows that V *is totally ramified in* L^*, i.e., it has a unique extension to L^* and its reduced ramification exponent equals $[L^*:K]$.

Let $[K(\zeta_u):K]=u^*$. Then $u^*<u$ and $K(\zeta_u)/K$ is an abelian Galois extension. Since V is totally ramified in L^* and the Y-discriminant of the polynomial Y^u-1 is a unit in R, we see that the fields L^* and $K(\zeta_u)$ are linearly disjoint over K. Therefore $L^*(\zeta_u)/K(\zeta_u)$ and $L^*(\zeta_u)/L^*$ are cyclic and abelian Galois extensions of degree u and u^* respectively. Let $R_1,R_2,\ldots,R_h$ be the distinct extensions of R to $K(\zeta_u)$, and let $[(R/M(R))(\zeta'_u):(R/M(R))]=u'$. Then $hu'=u^*$ and for $1\leqslant i\leqslant h$ we have that R_i is unramified over R and $[R_i/M(R_i):R/M(R)]=u'$, and hence in particular R_i is a d-dimensional regular local domain. Let $S^*=S\cap L^*$. Then S^* is the unique extension of R to L^*. Since S^* is residually rational over R, it follows that S^* has exactly h distinct extensions $S^*_1,S^*_2,\ldots,S^*_h$ to $L^*(\zeta_u)$, and for $1\leqslant i\leqslant h$ we have that S^*_i is unramified over S^* and $[S^*_i/M(S^*_i):S/M(S)]=u'$. Since S^* is integral over R, the local rings $S^*_1,S^*_2,\ldots,S^*_h$ can be labelled so that S^*_i is integral and residually rational

over R_i for $1 \leqslant i \leqslant h$. Since $L^*(\zeta_u)/K(\zeta_u)$ is cyclic of degree u, we can write $L^*(\zeta_u) = K(\zeta_u)(y)$ with $0 \neq y^u \in K(\zeta_u)$. Since R_1 is a UFD, upon dividing y by a suitable element in R_1 we can arrange matters so that $y^u = \delta y_1^{b_1} y_2^{b_2} \cdots y_a^{b_a}$ where b_i is a positive integer nondivisible by u for $1 \leqslant i \leqslant a$, the ideals $y_1 R_1, y_2 R_1, \ldots, y_a R_1$ are pairwise distinct prime ideals in R_1, and δ is a unit in R_1. Since V is ramified in L^*, we must have $a \neq 0$ and upon labelling the elements $y_1, y_2, \ldots, y_a$ suitably we can arrange matters so that V be the localization of R at the prime ideal $(y_1 R_1) \cap R$. Since V is totally ramified in L^*, *we must have* $\mathrm{GCD}(b_1, u) = 1$.

Now the Y-discriminant of the polynomial $Y^u - 1$ is a unit in R as well as S, and hence the sum of the separable parts of the residue degrees of the various extensions of R to $K(\zeta_u)$ (resp: S to $L(\zeta_u)$) equals the field degree $[K(\zeta_u):K]$ (resp: $[L(\zeta_u):L]$); since S is the unique extension of R to L and it is residually rational over R, by equating the double sum from R to $L(\zeta_u)$ via L with the double sum from R to $L(\zeta_u)$ via $K(\zeta_u)$ we see that $[L(\zeta_u):L] \geqslant [K(\zeta_u):K]$; therefore $[L(\zeta_u):L] = [K(\zeta_u):K]$ and the fields L and $K(\zeta_u)$ must be linearly disjoint over K; (this also gives an alternative proof of linear disjointness in the above two situations). It follows that $L(\zeta_u)/L^*(\zeta_u)$ is a finite algebraic field extension and $L^*(\zeta_u)/K(\zeta_u)$ is a cyclic Galois extension of degree u which is coprime to the degree of $L(\zeta_u)/L^*(\zeta_u)$. Moreover, S has exactly h distinct extensions $S_1, S_2, \ldots, S_h$ to $L(\zeta_u)$, and for $1 \leqslant i \leqslant h$ we have that S_i is unramified over S and $[S_i/M(S_i):S/M(S)] = u'$, and hence in particular S_i is a d-dimensional regular local domain. Finally, the local rings $S_1, S_2, \ldots, S_h$ can be labelled so that S_i is integral and residually rational over S_i^* for $1 \leqslant i \leqslant h$.

Let $\hat{R}_1$, $\hat{S}_1^*$ and $\hat{S}_1$ be the respective completions of R_1, S_1^* and S_1. Then $\hat{R}_1$ and $\hat{S}_1$ are d-dimensional regular local domains, and hence they are integrally closed in their respective quotient fields $\hat{K}_1$ and $\hat{L}_1$. In view of (37.8) on page 140 of [Nag] we see that $\hat{S}_1^*$ is also a d-dimensional local domain which is integrally closed in its quotient field $\hat{L}_1^*$. In view of (10.13) on page 243 of [A08] we may assume that $\hat{R}_1 \subset \hat{S}_1^* \subset \hat{S}_1$ and $\hat{K}_1 \subset \hat{L}_1^* \subset \hat{L}_1$, and then in view of standard properties of completions as given in §2 to §6 of Chapter VIII of [ZSa] we see that: $\hat{L}_1/\hat{L}_1^*$ is a finite algebraic field extension with $[\hat{L}_1:\hat{L}_1^*] = [L:L^*]$; $\hat{L}_1^*/\hat{K}_1$ is a cyclic Galois extension of degree $u = w^e$ which is coprime to $[\hat{L}_1:\hat{L}_1^*]$; e is a positive integer and w is a prime number which is nondivisible by the characteristic of $\hat{R}_1/M(\hat{R}_1)$; $\hat{S}_1^*$ and $\hat{S}_1$ are the unique extensions of $\hat{R}_1$ to $\hat{L}_1^*$ and $\hat{L}_1$ respectively; $\hat{S}_1$ is residually rational over $\hat{R}_1$; and $\hat{L}_1^* = \hat{K}_1(y)$. By (36.4) and (36.5) on page 132 of [Nag], for $1 \leqslant i \leqslant a$ we have $y_i = \delta_i y_{i1} y_{i2} \cdots y_{iv_i}$ where v_i is a positive integer, the ideals $y_{i1}\hat{R}_1, y_{i2}\hat{R}_1, \ldots, y_{iv_i}\hat{R}_1$ are pairwise distinct prime ideals in $\hat{R}_1$, and δ_i is a unit in $\hat{R}_1$. Taking a factorization of y in the UFD $\hat{S}_1$, raising it to the uth power, and then comparing the two factorizations of y^u in $\hat{R}_1$, we get $\xi_i^u = \rho_i y_{i1}^{b_i}$ with $\xi_i \in \hat{S}_1$ and $\rho_i \in \hat{S}_1 \backslash M(\hat{S}_1)$ for $1 \leqslant i \leqslant a$. Since $\hat{S}_1$ is residually rational over $\hat{R}_1$, we can find a unit σ_i in $\hat{R}_1$ with $\rho_i - \sigma_i \in M(\hat{S}_1)$ for $1 \leqslant i \leqslant a$. For $1 \leqslant i \leqslant a$ we can take an element τ_i in an overfield of $\hat{L}_1$ such that $\tau_i^u = \sigma_i$.

Let $\check{K}_1 = \hat{K}_1(\tau_1, \ldots, \tau_a)$, $\check{L}_1^* = \hat{L}_1^*(\tau_1, \ldots, \tau_a)$ and $\check{L}_1 = \hat{L}_1(\tau_1, \ldots, \tau_a)$. Let $\check{R}_1$ and $\check{S}_1$ be the integral closures of $\hat{R}_1$ and $\hat{S}_1$ in $\check{K}_1$ and $\check{L}_1$ respectively. Now the Y-discriminants of the polynomials $Y^u - \sigma_1, \ldots, Y^u - \sigma_a$ are units in $\hat{R}_1$, the local rings $\hat{R}_1$ and $\hat{S}_1$ are d-dimensional complete regular local domains, and $\hat{S}_1$ is residually rational over $\hat{R}_1$; therefore: the fields $\hat{L}_1$ and $\check{K}_1$ are linearly disjoint over the field $\hat{K}_1$; $\check{L}_1/\check{L}_1^*$ is a finite algebraic field extension with $[\check{L}_1:\check{L}_1^*] = [\hat{L}_1:\hat{L}_1^*]$; $\check{L}_1^*/\check{K}_1$ is a cyclic Galois extension of degree $u = w^e$ which is coprime to $[\check{L}_1:\check{L}_1^*]$; e is a positive integer and w is a prime number which is nondivisible by the characteristic of $\check{R}_1/M(\check{R}_1)$; $\check{R}_1$

and $\check{S}_1$ are d-dimensional regular local domains with respective quotient fields $\check{K}_1$ and $\check{L}_1$ such that $\check{S}_1$ dominates and is integral and residually rational over $\check{R}_1$; the ring $\check{R}_1$ contains ζ_u which is a primitive uth root of 1; and $y_{11}\check{R}_1, y_{21}\check{R}_1, \ldots, y_{a1}\check{R}_1$ are pairwise distinct prime ideals in $\check{R}_1$. Upon letting $\omega_1 : \check{S}_1 \rightarrow \check{S}_1/M(\check{S}_1)$ to be the canonical epimorphism, for $1 \leqslant i \leqslant a$, the polynomial $Y^u - \omega_1(\rho_i)$ completely factors in $\omega_1(\check{S}_1)[Y]$ into pairwise distinct monic linear factors and hence by Hensel's Lemma we get $\mu_i^u = \rho_i$ for some $\mu_i \in \check{S}_1$; therefore upon letting $\eta_i = \xi_i/\mu_i$ we obtain $\eta_i^u = y_{i1}^{b_i}$ with $\eta_i \in \check{S}_1$; since $u/[\mathrm{GCD}(u, b_i)] \neq 1 = \mathrm{GCD}(u, [\check{L}_1 : \check{L}_1^*])$, we must have $\eta_i \in \check{L}_1^*$. Upon letting $\lambda = \Pi_{1 \leqslant i \leqslant a} u/[\mathrm{GCD}(u, b_i)]$ we clearly have: $a \neq 1 \Leftrightarrow \lambda > u$. By considering ramification of the DVRs obtained by localizing $\check{R}_1$ at the prime ideals $y_{11}\check{R}_1, y_{21}\check{R}_1, \ldots, y_{a1}\check{R}_1$, we see that $[\check{K}_1(\eta_1, \ldots, \eta_a) : \check{K}_1] = \lambda$. Since $[\check{L}_1^* : \check{K}_1] = u$, *we must have* $a = 1$. *Therefore V is the only DVR in $\mathfrak{B}(R)$ which is ramified in L^*.*

Now it only remains to show that, assuming V to have a simple point at R and given any field K^* with $K \subset K^* \subset L^*$, the local ring $S \cap K^*$ is regular. Since $\mathrm{GCD}(u, b_1) = 1$, we can find positive integers b_0 and u_0 such that $b_0 b_1 - u_0 u = 1$, and upon letting $y^* = y^{b_0}/y_1^{u_0}$ we get $L^* = K(y^*)$ and $y^{*u} = \delta^* y_1$ with $\delta^* \in R_1 \backslash M(R_1)$. Let $[K^* : K] = \bar{u}$ and $\bar{y} = y^{*u/\bar{u}}$. Then $\bar{u} = w^{\bar{e}}$ with nonnegative integer $\bar{e} \leqslant e$ and we have $K^*(\zeta_u) = K(\zeta_u)(\bar{y})$ with $\bar{y}^{\bar{u}} = \delta^* y_1$. Let $R^* = S \cap K^*$ and $R_1^* = S_1^* \cap K^*(\zeta_u)$. Then R^* and R_1^* are the unique extensions of R and R_1 to K^* and $K^*(\zeta_u)$ respectively, R^* is unramified in $K^*(\zeta_u)$, and R_1^* is an extension of R_1^* to $K^*(\zeta_u)$. Since V has a simple point at R, we must have $y_1 \in M(R_1) \backslash M(R_1)^2$, and hence R_1^* is a d-dimensional regular local domain. Therefore the completion $\hat{R}_1^*$ of R_1^* is a d-dimensional regular local domain, and hence in particular $\hat{R}_1^*$ is integrally closed in its quotient field $\hat{K}_1^*$. In view of (37.8) on page 140 of [Nag] we see that $\hat{R}^*$ is also a d-dimensional local domain which is integrally closed in its quotient field $\hat{K}^*$. In view of (10.13) on page 243 of [A08] we may assume that $\hat{R}^* \subset \hat{R}_1^*$ and $\hat{K}^* \subset \hat{K}_1^*$, and then in view of standard properties of completions as given in §2 to §6 of Chapter VIII of [ZSa] we see that: $\hat{K}_1^*/\hat{K}^*$ is a finite algebraic separable field extension, $\hat{R}_1^*$ is the unique extension of $\hat{R}^*$ to $\hat{K}_1^*$, $\hat{R}^*$ is unramified in $\hat{K}_1^*$, and $\hat{R}_1^*$ is regular. Now by Proposition 26 on page 12 of [A06] we conclude that $\hat{R}^*$ is regular. Therefore R^* is regular, i.e., $S \cap K^*$ *is regular.*

Lemma 1.2. *Let $K \subset L^* \subset L$ be fields such that L/L^* is a finite algebraic extension and L^*/K is a cyclic Galois extension whose degree is coprime to the degree of L/L^*. Let d be a positive integer and let R and S be d-dimensional pseudogeometric regular local domains with quotient fields K and L, respectively, such that S is integral and residually rational over R. Assume that $L^* \neq K$ and $[L^* : K]$ is nondivisible by the characteristic of $R/M(R)$. Then there is at least one and at most a finite number of (pairwise distinct) DVRs $V_1, V_2, \ldots, V_m$ in $\mathfrak{B}(R)$ which are ramified in L^*. Moreover, upon letting r_i to be the reduced ramification exponent of an extension of V_i to L^*, we have that the integers $r_1, r_2, \ldots, r_m$ are pairwise coprime and their product $r_1 r_2 \cdots r_m$ equals $[L^* : K]$. Finally, if for every field K^* with $K \subset K^* \subset L^*$ for which $S \cap K^*$ is a regular local domain, and for every DVR V in $\mathfrak{B}(S \cap K^*)$ which is ramified in L^*, we have that V has a simple point at $S \cap K^*$, then for every field K^* with $K \subset K^* \subset L^*$ we have that $S \cap K^*$ is a d-dimensional regular local domain, and if also $L \neq L^*$ then some DVR in $\mathfrak{B}(S \cap L^*)$ must be ramified in L.*

Proof. We can take a factorization $[L^* : K] = w_1^{e_1} w_2^{e_2} \cdots w_\mu^{e_\mu}$ with positive integers $e_1, e_2, \ldots, e_\mu$ and pairwise distinct prime numbers $w_1, w_2, \ldots, w_\mu$. Since L^*/K is cyclic, for $1 \leqslant \nu \leqslant \mu$, there is a unique field L_ν^* with $K \subset L_\nu^* \subset L^*$ such that $[L_\nu^*:K] = w_\nu^{e_\nu}$. By Lemma (1.1) we have a surjective map ψ of the integral segment $[1,\mu]$ onto the integral segment $[1,m]$ such that, for $1 \leqslant \nu \leqslant \mu$, the DVR $V_{\psi(\nu)}$ is totally ramified in L_ν^* and the DVR V_i is unramified in L_ν^* for every $i \in [1,m] \backslash \{\psi(\nu)\}$. Obviously $r_i = \Pi_{\nu \in \psi^{-1}(i)} w_\nu^{e_\nu}$ for $1 \leqslant i \leqslant m$, and hence the integers $r_1, r_2, \ldots, r_m$ are pairwise coprime and their product $r_1 r_2, \ldots, r_m$ equals $[L^*:K]$. By induction on the sum $e_1 + e_2 + \cdots + e_\mu$, we shall now prove the *claim* that if: (*) for every field K^* with $K \subset K^* \subset L^*$ for which $S \cap K^*$ is a regular local domain, and for every DVR V in $\mathfrak{V}(S \cap K^*)$ which is ramified in L^*, we have that V has a simple point at $S \cap K^*$, then: (′) for every field K^* with $K \subset K^* \subset L^*$ we have that $S \cap K^*$ is a d-dimensional regular local domain. If the sum is 1 then the claim follows from Lemma 1.1. So let the sum be at least 2 and assume that the claim is true for all values of the sum smaller than the given value. Also assume (*) and let there be given any field $K \subset K^* \subset L^*$. Now if $K^* \neq L^*$ then by the induction hypothesis it follows that $S \cap K^*$ is a d-dimensional regular local domain. On the other hand, if $K^* = L^*$ then we can take a field K_0 with $K \subset K_0 \subset K^*$ such that $[K^*:K_0]$ is a prime number, and now by the induction hypothesis we see that $S \cap K_0$ is a d-dimensional regular local domain and hence by Lemma 1.1 we conclude that $S \cap K^*$ is a d-dimensional regular local domain. This completes the induction and hence proves the claim. Finally, continue to assume (*) and also assume that $L \neq L^*$; now by the claim we know that $S \cap L^*$ is regular and hence by the *purity of branch locus* (see Theorem 41.1 on page 158 of [Nag]) we conclude that some DVR in $\mathfrak{V}(S \cap L^*)$ must be ramified in L.

Lemma 1.3. *Let R be a 2-dimensional local domain dominating a DVR T such that $R/M(T)R$ is regular. Then R is regular and upon letting U to be the DVR in $\mathfrak{V}(R)$ obtained by localizing R at $M(T)R$ we have that U has a simple point at R, and for every DVR V in $\mathfrak{V}(R) \backslash \{U\}$ which is residually rational over T we have that $\{U, V\}$ has a normal crossing at R and hence in particular V has a simple point at R. Morever, if there does exist a DVR in $\mathfrak{V}(R) \backslash \{U\}$ which is residually rational over T, then R must be residually rational over T.*

Proof. Since T is a DVR, we have $M(T) = tT$ for some $0 \neq t \in T$. By assumption R is a 2-dimensional local domain dominating T such that $R/M(T)R$ is regular; therefore $R/M(T)R$ must be a 1-dimensional regular local domain and hence its maximal ideal is generated by the image of some $0 \neq y \in M(R)$. Clearly $M(R) = (y,t)R$ and hence R is regular; also $M(U) \cap R = M(T)$ and hence U has a simple point at R. Given any DVR V in $\mathfrak{V}(R) \backslash \{U\}$, we have $M(V) \cap R = zR$ for some $z \in M(R) \backslash (tR)$. Now assume that V is residually rational over T. Then, for any $\eta \in R$ we can find η' in the quotient field of T such that $\eta - \eta' \in M(V)$. If $\eta' \notin T$ then there is a positive integer b such that $t^b \eta' \in T \backslash M(T)$, and this gives $t^b \eta - t^b \eta' \in M(V) \cap R \subset M(R)$ which is a contradiction because $t^b \eta \in M(R)$ and $t^b \eta' \notin M(R)$. Therefore $\eta' \in T$ and hence $\eta - \eta' \in M(V) \cap R = (zR) \subset M(R)$; It follows that if $\eta \in M(R)$ then $\eta' \in M(R) \cap T = (tR)$; consequently $(z,t)R = M(R)$ and hence $\{U, V\}$ have a normal crossing at R and V has a simple point at R. Since for every $\eta \in R$ we have $\eta - \eta' \in M(R)$ with $\eta' \in \mathrm{T}$, we also see that R is residually rational over T.

PROPOSITION 1.4.

Let R be a 2-dimensional regular local domain dominating a DVR T such that $R/M(T)R$ is regular, and let U be the DVR in $\mathfrak{V}(R)$ obtained by localizing R at $M(T)R$. Let B

be the integral closure of R *in a Galois extension* L *of the quotient field* K *of* R. *Assume that* $B/M(T)B$ *is a regular domain, and let* Θ *be the localization of* B *at the prime ideal* $M(T)B$. *[Clearly then:* Θ *is a DVR which is the integral closure of* U *in* L, *and* U *is unsplit and naively unramified in* L.*] Let* $\tilde{G}$ *and* $\tilde{L}$ *respectively be the inertia group and the inertia field of* Θ *over* U, *let* $\phi:\Theta \to L' = \Theta/M(\Theta)$ *be the canonical epimorphism, let* $K' = \phi(U)$ and $\tilde{L}' = \phi(\tilde{\Theta})$ *with* $\tilde{\Theta} = \Theta \cap \tilde{L}$, *and let* $k' = \phi(T)$ *and* $R' = \phi(R)$ *and* $B' = \phi(B)$.

[Clearly then (see Chapter 1 of [A04] and Chapter V of [ZSa]): $M(\Theta) \cap U = M(U)$ and $K' \approx U/M(U)$; $M(\Theta) \cap T = M(T)$ and $k' \approx T/M(T)$; $M(\Theta) \cap R = M(T)R$ and $R' \approx R/M(T)R$; $M(\Theta) \cap B = M(T)B$ and $B' \approx B/M(T)B$; $\tilde{G}$ is a normal subgroup of $\mathrm{Gal}(L, K)$ and L is the fixed field of G; the order of $\tilde{G}$ equals 1 or a nonnegative power of the characteristic of k' according as the said characteristic is zero or not; U is unramified in $\tilde{L}$; L'/K' is a finite normal algebraic field extension and ϕ induces an epimorphism of $\mathrm{Gal}(L, K)$ onto $\mathrm{Gal}(L', K')$ with kernel $\tilde{G}$; $L'/\tilde{L}'$ is purely inseparable with $[L':\tilde{L}'] = [L:\tilde{L}] = |\tilde{G}|$; $\tilde{L}'/K'$ is a Galois extension with $\mathrm{Gal}(\tilde{L}', K') \approx \mathrm{Gal}(L', K')$; R' is a DVR with quotient field K' and subfield k'; and B' is the integral closure of R' in L'.]

Assume that every extension to L *of any DVR in* $\mathfrak{V}(R)\backslash\{U\}$, *which is ramified in* L, *is residually rational over* T. *Let* $V_1, V_2, \ldots, V_m$ *be all the distinct DVRs in* $\mathfrak{V}(R)\backslash\{U\}$ *which are ramified in* L. *For* $1 \leqslant i \leqslant m$, *let* r_i *be the ramification exponent of an extension of* V_i *to* L, *and let* s_i *be the part of* r_i *prime to the residue characteristic of* T. *Let* r *be the reduced ramification exponent of an extension of* R' *to* L', *and let* s *be the part of* r *prime to the residue characteristic of* T. *Then we have the following.*

If $m = 0$ *then:* R' *is unramified in* $\tilde{L}'$. *If* $m > 0$ *then:* $s_1, s_2, \ldots, s_m$ *are pairwise coprime, and their product* $s_1, s_2, \ldots, s_m$ *equals* s. *If* $m > 0$ *then:* R *is residually rational over* T, *and* V_i *has a simple point at* R *for* $1 \leqslant i \leqslant m$. *If* $m > 0$ *then:* R' *is residually rational over* k', *and* $r/s \geqslant r_i/s_i$, *for* $1 \leqslant i \leqslant m$. *Finally, if* $m > 0$ *then:* $r = s \Leftrightarrow L = \tilde{L}$ *and* $r_i = s_i$ *for* $1 \leqslant i \leqslant m$.

Proof. Let $S_1, S_2, \ldots, S_h$ be all the distinct extensions of R to L, i.e., the localizations of B at its various maximal ideals. Note that $M(T)B \subset M(S_j) \cap B$ for $1 \leqslant j \leqslant h$ and let S'_j be the localization of B' at its maximal ideal $\phi(M(S_j \cap B))$. Then clearly $S'_1, S'_2, \ldots, S'_h$ are exactly all the distinct extensions of R' to L'. Let $\tilde{S}'_j = S'_j \cap \tilde{L}'$ for $1 \leqslant j \leqslant h$. Then $\tilde{S}'_1, \tilde{S}'_2, \ldots, \tilde{S}'_h$ are exactly all the distinct extensions of R' to $\tilde{L}'$, and for $1 \leqslant j \leqslant h$ we have that S'_j is the unique extension of $\tilde{S}'_j$ to L'.

For $1 \leqslant j \leqslant h$, let G_j^{σ} and G_j^{τ} be the splitting and inertia groups of S_j over R and note that then G_j^{σ} is a subgroup of $\mathrm{Gal}(L, K)$ and G_j^{τ} is a normal subgroup of G_j^{σ}, and let K_j^{σ} and K_j^{τ} be the splitting and inertia fields of S_j over R and note these are the fixed fields of G_j^{σ} and G_j^{τ} respectively. Let $\phi_{\mathrm{gal}}:\mathrm{Gal}(L, K) \to \mathrm{Gal}(L', K')$ and $\tilde{\phi}_{\mathrm{gal}}:\mathrm{Gal}(L, K) \to \mathrm{Gal}(\tilde{L}', K')$ be the epimorphisms induced by ϕ, and note that for every $g \in \mathrm{Gal}(L, K)$ we have $\tilde{\phi}_{\mathrm{gal}}(g) = \phi_{\mathrm{gal}}(g)|\tilde{L}'$, and for $1 \leqslant j \leqslant h$ we have $\ker(\phi_{\mathrm{gal}}) = \ker(\tilde{\phi}_{\mathrm{gal}}) = \tilde{G} \subset G_j^{\tau}$. Again for $1 \leqslant j \leqslant h$, let $\tilde{G}_j'^{\sigma}$ and $\tilde{G}_j'^{\tau}$ be the splitting and inertia groups of $\tilde{S}'_j$ over R' and note that then $\tilde{G}_j'^{\sigma}$ is a subgroup of $\mathrm{Gal}(\tilde{L}', K')$ and $\tilde{G}_j'^{\tau}$ is a normal subgroup of $\tilde{G}_j'^{\sigma}$, and let $\tilde{K}_j'^{\sigma}$ and $\tilde{K}_j'^{\tau}$ be the splitting and inertia fields of $\tilde{S}'_j$ over R' and note these are the fixed fields of $\tilde{G}_j'^{\sigma}$ and $\tilde{G}_j'^{\tau}$ respectively. Now clearly $\tilde{\phi}_{\mathrm{gal}}(G_j^{\sigma}) = \tilde{G}_j'^{\sigma}$ and $\tilde{\phi}_{\mathrm{gal}}(G_j^{\tau}) = \tilde{G}_j'^{\tau}$ for $1 \leqslant j \leqslant h$.

Since U is unramified in $\tilde{L}$, in view of (10.13) of [A08] and (37.8) and (41.4) of [Nag] and §2 to §6 of Chapter VIII [ZSa], we see that if $m = 0$ then $K_j^{\tau} = \tilde{L}$ for $1 \leqslant j \leqslant h$ and hence $\tilde{G}_j'^{\tau} =$ for $1 \leqslant j \leqslant h$, and therefore R' is unramified in $\tilde{L}'$.

So henceforth assume that $m>0$. Then in view of Lemma (1.3) we see that $S_1, S_2, \ldots, S_h$ are residually rational over T, and hence R and R' are residually rational over T and k' respectively, and $S_1, S_2, \ldots, S_h$ are residually rational over R, and therefore $G_j^\sigma = G_j^\tau$ for $1 \leqslant j \leqslant h$, and hence $\tilde{G}_j'^\sigma = \tilde{G}_j'^\tau$ for $1 \leqslant j \leqslant h$. By Lemma (1.3) we also see that V_i has a simple point at R for $1 \leqslant i \leqslant m$.

Since R' is a DVR, for $1 \leqslant j \leqslant h$, the group $\tilde{G}_j'^\tau$ has a unique (normal) p-Sylow subgroup $\tilde{G}_j'^*$ where p is the residue characteristic of T (take $\tilde{G}_j'^* = 1$ in case the residue characteristic of T is zero) and the factor group $\tilde{G}_j'^\tau/\tilde{G}_j'^*$ is cyclic; now upon letting $\tilde{K}_j'^*$ to be the fixed field of $\tilde{G}_j'^*$ we clearly have $[L' : \tilde{K}_j'^\tau] = r$ and $[\tilde{L}_j'^* : \tilde{K}_j'^\tau] = s$, and hence upon letting L_j^* to be the fixed field of $\tilde{\phi}_{\text{gal}}^{-1}(\tilde{G}_j'^*)$ we see that: $K \subset K_j^\sigma = K_j^\tau \subset L_j^* \subset \tilde{L} \subset L$, the extension L_j^*/K_j^τ is a cyclic Galois extension of degree s which is nondivisible by the residue characteristic of T, and the extension L/L_j^* is a Galois extension of degree r/s where r/s equals 1 or a nonnegative power of the residue characteristic of T according as the said characteristic is zero or not.

Upon letting $R_1^\tau = S_1 \cap K_1^\tau$ we have that R and R_1^τ are d-dimensional regular local domains such that R_1^τ is unramified and residually rational over R; for $1 \leqslant i \leqslant m$, we know that V_i has a simple point at R and every extension of V_i to L is residually rational over T, and hence V_i has a unique extension V_i^τ to K_1^τ which belongs to $\mathfrak{B}(R_j^\tau)$ and this unique extension V_i^τ is unramified and residually rational over V_i, and therefore r_i is the ramification exponent of every extension of V_i^τ to L, and s_i is the part of r_i prime to the residue characteristic of T. Upon letting $U_1^\tau = \Theta \cap K_1^\tau$ we see that U_1^τ is a DVR in $\mathfrak{B}(R_1^\tau)$, and $V_1^\tau, V_2^\tau, \ldots, V_m^\tau$ are exactly all the DVRs in $\mathfrak{B}(R_1^\tau) \backslash \{U_1^\tau\}$ which are ramified in L; since U is unramified in L, we see that U_1^τ is unramified in L_j^*.

Upon relabelling $V_1, V_2, \ldots, V_m$ suitably, we can arrange matters so that $V_1^\tau, V_2^\tau, \ldots, V_n^\tau$ are ramified in L_1^* and $V_{n+1}^\tau, V_{n+2}^\tau, \ldots, V_m^\tau$ are unramified in L_1^*. Now clearly $s_{n+1} = s_{n+2} = \cdots = s_m = 1$, and for $1 \leqslant i \leqslant n$ we have that s_i is the ramification exponent of every extension of V_i^τ to L_1^*, and for $1 \leqslant i \leqslant m$ we have that r_i/s_i is the ramification exponent of every extension of V_i^τ to L and hence $r/s \equiv 0(r_i/s_i)$.

If $L_1^* = K_1^\tau$ then $s=1$ and $n=0$, and hence obviously: the local ring $S_1 \cap R_1^*$ is regular, the integers $s_1, s_2, \ldots, s_m$ are pairwise coprime, and their product $s_1 s_2 \cdots s_m$ equals s. If $L_1^* \neq K_1^\tau$ then by taking $(K_1^\tau, L_1^*, L, R_1^\tau, S_1, n, V_1^\tau, V_2^\tau, \ldots, V_n^\tau, s_1, s_2, \ldots, s_n)$ for $(K, L^*, L, R, S, m, V_1, V_2, \ldots, V_m, r_1, r_2, \ldots, r_m)$ in Lemma (1.2) we see that: $n>0$, the local ring $S_1 \cap L_1^*$ is regular, the integers $s_1, s_2, \ldots, s_m$ are pairwise coprime, and their product $s_1 s_2 \cdots s_m$ equals s.

Now it only remains to note that if $L = \tilde{L}$ and $r_i = s_i$ for $1 \leqslant i \leqslant m$ then $S_1 \cap L_1^*$ is a regular local domain with quotient field L_1^*, S_1 is the unique extension if $S_1 \cap L_1^*$ to L, S_1 is residually rational over $S_1 \cap L_1^*$, and every DVR in $\mathfrak{B}(S_1 \cap L_1^*)$ is unramified in L, and hence, in view of (10.13) of [A08] and (41.4) of [Nag] and §2 to §6 of Chapter VIII [ZSa], we must have $L = L_1^*$ and therefore $r = s$.

The following proposition about divisors of second kind is proved in [A01].

PROPOSITION 1.5.

Let U be a prime divisor of second kind of a 2-dimensional regular local domain R and let $R = R_0 \subset R_1 \subset R_2 \subset \cdots$ be the quadratic sequence along U. Then the residue field of U is a simple transcendental extension of a finite algebraic extension of the residue field of R and there is a unique positive integer s such that $R_0 \neq R_1 \neq R_2 \cdots \neq R_s = U = R_{s+1} = R_{s+2} = \ldots$. Moreover, $\dim R_i = 2$ for $1 \leqslant i \leqslant s$, and $\mathrm{ord}_U(z) = \mathrm{ord}_{R_{s-1}}(z)$

for every element z in the quotient field of R. Finally, if $M(R)=(x_0,t)R$ *with* $\mathrm{ord}_U(t)=1$, *the residue field of R is relatively algebraically closed in the residue field of U, and* $\mathfrak{k}$ *is a coefficient set for R, then there exist unique elements* $c_1,c_2,\ldots,c_s$ *in* $\mathfrak{k}$ *with* $c_s=0$ *such that upon letting* $x_i=x_{i-1}t^{-1}-c_i$ *for* $1\leqslant i\leqslant s$ *we have* $M(R_i)=(x_i,t)R_i$ *for* $1\leqslant i\leqslant s$, *and* x_s *is a residual transcendental generator of U over R.*

As a consequence of the above Proposition (1.5) we have the following.

COROLLARY 1.6

Let U be a DVR dominating another DVR T such that: the quotient field K of U is a simple transcendental extension of the quotient field k^* *of T, the residue field of U is transcendental over the residue field of T, the residue field of T is relatively algebraically closed in the residue field of U, and* $M(U)=M(T)U$. *Then* $K=k^*(x)$ *for some x which is a residual transcendental generator of U over T. Moreover, for every such x we have that U is the localization of* $T[x]$ *at the prime ideal* $M(T)\ T[x]$.

Namely, by assumption $M(T)=tT$ for some t, and $K=k^*(x^*)$ for some x^*. Now $\mathrm{ord}_U(t)=1$, and we can find an integer e such that $e+\mathrm{ord}_U(x^*)>0$. Let $x_0=t^ex^*$, let R be the localization of $T[x_0]$ at the prime ideal $(x_0,t)\ T[x_0]$, and let $\mathfrak{k}$ be a coefficient set for T. Then U is a prime divisor of second kind of the 2-dimensional regular local domain R, and $\mathfrak{k}$ is a coefficient set for R, and hence it suffices to take $x=x_s$ with x_s as in the above Proposition (1.5).

2. Specializations of coverings of the line

Referring to the preamble to § 1 for geometric outline, we want to compare the Galois theories of the generic and special fibers of an arithmetic surface over a discrete valuation ring. As remarked before, the correspondence between these two fibers is infinite to one, which causes some difficulty in showing that different automorphisms of the generic fiber induce different automorphisms of the special fiber. Assuming the genus to be at least two, this is overcome by passing to points of finite order of the Jacobian varieties of the two fibers. In precise algebraic terms we proceed thus.

Let L/k^* *be a* 1*-dimensional function field*, i.e., L is a field which is finitely generated and of transcendence degree 1 over the subfield k^*. Recall that for any subring $\bar{k}$ of any field $\bar{L}$, the *Riemann–Zariski space* $\mathfrak{R}(\bar{L}/\bar{k})$ is the set of all valuation rings of $\bar{L}$ which contain $\bar{k}$. In our case, $L\in\mathfrak{R}(L/k^*)$ and the members of $\mathfrak{R}(L/k^*)\backslash\{L\}$ are DVRs which are in 1–1 correspondence with the points of a nonsingular projective algebraic curve over k^*. More precisely, $\mathfrak{R}(L/k^*)$ is the unique nonsingular projective model of L/k^*; L is its generic point and all others are its closed points. So we may visualize $\mathfrak{R}(L/k^*)$ as a (nonsingular projective algebraic) *curve*. Eventually we shall assume that L/k^* is *regular* (as a function field), i.e., k^* is relatively algebraically closed in L and L is separably generated over k^*.

Let T be a pseudogeometric DVR *with quotient field* k^*; note that a DVR of characteristic zero is automatically pseudogeometric. Eventually we shall assume that the residue field of T is perfect; note that a finitely generated field extension of a perfect field is automatically separably generated, and also note that a field of characteristic zero is always perfect.

Let E be a projective model of L/T; for definition see page 27 of [A08]. Now

$\dim E = 2$ and hence we may think of E as an *arithmetic surface*. Let E_T be the set of all those members of E which do not dominate T; obviously E_T is a projective model of L/k^* and we call it *the generic fiber of E over T*. We shall only be interested in the case when E_T is nonsingular; note that this is so iff $E_T = \mathfrak{R}(L/k^*)$. Clearly there is a unique ideal T_E on E such that for every $S \in E$ we have $T_E S = M(T)S$ and for every affine ring A over T with $\mathfrak{V}(A) \subset E$ we have $A \cap T_E = M(T)A$; we call T_E *the ideal induced by T on E*; for the definition of an ideal on a model see page 169 of [A08]. Note that E is the disjoint union of E_T and the zero-set $\mathfrak{Z}(T_E)$. The zero-set $\mathfrak{Z}(T_E)$ together with the ideal T_E may be called the *special fiber of E at T*; since we are only interested in the case when the special fiber is *reduced*, i.e., when its ideal $\mathfrak{I}(\mathfrak{Z}(T_E), E)$ on E is equal to T_E, we may simply *refer to $\mathfrak{Z}(T_E)$ as the special fiber*. Note that if $\mathfrak{Z}(T_E)$ is irreducible then its generic point is 1-dimensional and its remaining points are exactly all the 2-dimensional members of E.

Henceforth assume that E is good (as a model of L/T), by which we mean that $\mathfrak{Z}(T_E)$ is irreducible and nonsingular and for its ideal on E we have $\mathfrak{I}(\mathfrak{Z}(T_E), E) = T_E$. In view of (1.3) we see that E is nonsingular, and hence in particular $E_T = \mathfrak{R}(L/k^*)$ and the generic point Θ of $\mathfrak{Z}(T_E)$ is a DVR dominating T. Also note that every $S \in \mathfrak{Z}(T_E) \setminus \{\Theta\}$ is the intersection of all the DVRs in $\mathfrak{V}(S)$. Let $\phi: \Theta \to L' = \Theta/M(\Theta)$ be the canonical epimorphism, and let $k' = \phi(T)$. Now clearly L'/k' is a 1-dimensional function field and $S \mapsto \phi(S)$ gives a bijection of $\mathfrak{Z}(T_E)$ onto $\mathfrak{R}(L'/k')$. Some of the things said so far may be depicted in the following diagram.

$$\begin{tikzcd}
L & \Theta \arrow[l] \arrow[r, "\phi"] & L' \\
k^* \arrow[u, dash] & T \arrow[l] \arrow[r, "\phi"] \arrow[u, dash] & k' \arrow[u, dash]
\end{tikzcd}$$

Now let $\mathrm{Div}(L/k^*)$ be the group of all divisors of L/k^* which we regard as functions $\mathfrak{R}(L/k^*) \setminus \{L\} \to \mathbb{Z}$ with finite support, let $\mathrm{Prin}(L/k^*) \subset \mathrm{Div}_0(L/k^*) \subset \mathrm{Div}(L/k^*)$ be the subgroups of principal divisors and divisors of degree zero respectively, and let $\mathrm{Jac}(L/k^*)$ be the factor group $\mathrm{Div}_0(L/k^*)/\mathrm{Prin}(L/k^*)$; likewise for L'/k'. By referring to [Lam] for details, for $W \in \mathfrak{R}(L/k^*) \setminus \{L\}$ we get $\phi_{\mathrm{val}}(W) \in \mathrm{Div}(L'/k')$ by putting $\phi_{\mathrm{val}}(W)(\phi(S)) = 0$ if $S \in \mathfrak{Z}(T_E) \setminus \{\Theta\}$ is such that $W \notin \mathfrak{V}(S)$, and $\phi_{\mathrm{val}}(W)(\phi(S)) = \mathrm{ord}_{\phi(S)} \phi(M(W) \cap S)$ if $S \in \mathfrak{Z}(T_E) \setminus \{\Theta\}$ is such that $W \in \mathfrak{V}(S)$, and we note that the degree of $\phi_{\mathrm{val}}(W)$ equals $[W/M(W) : k^*]$. In particular, for every $W \in \mathfrak{R}(L/k^*) \setminus \{L\}$ we have $W \in \mathfrak{V}(S)$ for some $S \in \mathfrak{Z}(T_E) \setminus \{\Theta\}$, and hence the members of $\mathfrak{Z}(T_E) \setminus \{\Theta\}$ are exactly all the closed points of E. By additivity this gives a degree preserving (group) homomorphism $\phi_{\mathrm{div}}: \mathrm{Div}(L/k^*) \to \mathrm{Div}(L'/k')$. It is easily seen that ϕ_{div} preserves principalness and hence it gives rise to a homomorphism $\phi_{\mathrm{jac}}: \mathrm{Jac}(L/k^*) \to \mathrm{Jac}(L'/k')$. For any positive integer v, this induces a homomorphism $\phi_{\mathrm{jac}}^{(v)}: \mathrm{Jac}^{(v)}(L/k^*) \to \mathrm{Jac}^{(v)}(L'/k')$ where $\mathrm{Jac}^{(v)}(L/k^*)$ and $\mathrm{Jac}^{(v)}(L'/k')$ are the respective subgroups of $\mathrm{Jac}(L/k^*)$ and $\mathrm{Jac}(L'/k')$ consisting all those points whose order divides v.

Henceforth assume that $T/M(T)$ is perfect, some member of $\mathfrak{R}(L/k^*)$ is residually rational over k^*, and L is separably generated over k^*; note that then L/k^* and L'/k' are regular function fields. Under these conditions, by a result of Hironaka [Hir] and Popp [Pop] we have the following.

PROPOSITION 2.1.

The genus of L'/k' equals the genus of L/k^*.

Henceforth assume that the genus of L/k^* is at least 2, and let K be a subfield of L such that L/K is Galois and K is a simple transcendental extension of k^*. Let $U = \Theta \cap K$, and note that then U is a DVR with quotient field K such that U dominates T and $M(T)U = M(U)$; also Θ is the unique extension of U to L, and Θ is naively unramified over U. *Let $K' = \phi(U)$ and note that then L'/K' is a finite normal field extension and L'/k' is a simple transcendental extension*; also we get an epimorphism $\phi_{\mathrm{gal}}:\mathrm{Gal}(L,K) \to \mathrm{Gal}(L',K')$ whose kernel is the inertia group $\underset{\sim}{G}$ of Θ over U and we have that: ϕ_{gal} is injective $\Leftrightarrow$ L'/K' is Galois. With all these assumptions, in Proposition (2.2) we shall now prove that we can find a good "common" coordinate function x for K and K' as indicated in the following diagram.

$$\begin{tikzcd}
L & \Theta \arrow[l] \arrow[r, "\phi"] & L' \\
K = k^*(x) \arrow[u, dash] & U = T[x] \arrow[u, dash] \arrow[l] \arrow[r, "\phi"] & K' = k'(\phi(x)) \arrow[u, dash] \\
k^* \arrow[u, dash] & T \arrow[u, dash] \arrow[l] \arrow[r, "\phi"] & k' \arrow[u, dash]
\end{tikzcd}$$

PROPOSITION 2.2.

There exists a residual transcendental generator x of U over T and, for any such x, the localization of $T[x]$ at the prime ideal $M(T)\,T[x]$ coincides with U. Moreover, upon letting D to be the projective line over T defined by x, i.e., upon letting D to be the model of K/T given by $D = \mathfrak{B}(T[x]) \cup \mathfrak{B}(T[1/x])$, we have that E is the normalization of D in L, i.e., E consists of all the extensions of the various members of D to L. (Therefore $D = E \cap K$ in the sense that: for any field extension $L^\flat/K^\flat$ and any set $E^\flat$ of subrings of $L^\flat$, by $E^\flat \cap K^\flat$ we may denote the set of subrings of $K^\flat$ given by $\{S^\flat \cap K^\flat : S^\flat \in E^\flat\}$).

Proof. By (1.6) there exists a residual transcendental generator x of U over T and, for any such x, the localization of $T[x]$ at the prime ideal $M(T)\,T[x]$ coincides with U, and hence upon letting D to be the projective line over T defined by any such x, i.e., upon letting D to be the model of K/T given by $D = \mathfrak{B}(T[x]) \cup \mathfrak{B}(T[1/x])$, we have $U \in D$. Let E^* be the normalization of D in L. Then clearly $(\mathfrak{R}(L/k^*) \cup \{\Theta\}) \subset E^*$ and $E^* \setminus (\mathfrak{R}(L/k^*) \cup \{\Theta\})$ is a set of 2-dimensional normal local domains. Also L' is the residue field of Θ and by (2.1) we know that the genus of L'/k' is at least two. Consequently by (1.5) we see that in the birational correspondence between E and E^* there can be no fundamental points on E, and hence E dominates E^*, and therefore by ZMT (= Zariski's Main Theorem) we must have $E = E^*$, (for the arithmetic versions of ZMT and the concept of fundamental points etc., see [A07] and [A08]).

With all the above assumptions in force, we shall now prove the following.

PROPOSITION 2.3

ϕ_{gal} *is injective.*

Proof. Let g be the genus of L/k^*, and hence by (2.1) also the genus of L'/K'. Let $\check{k}$ be an algebraic closure of k^*, let $\check{k}'$ be an algebraic closure of k', and let $\check{K} = K(\check{k})$, $\check{L} = L(\check{k})$, $\check{K}' = K'(\check{k}')$, and $\check{L}' = L'(\check{k}')$. Then the genus of $\check{L}/\check{k}$ as well as that of $\check{L}'/\check{k}'$

is g, and $\mathrm{Gal}(\check{L}, \check{K})$ and $\mathrm{Gal}(\check{L}', \check{K}')$ can be identified with $\mathrm{Gal}(L, K)$ and $\mathrm{Gal}(L', K')$ respectively. Let l be a prime number nondivisible by the characteristic of k', and let $v = l^m$ where m is a positive integer. Then by Weil's Theorem on points of finite order of abelian varieties (see Corollary 1 on page 127 of [Wei]) we have $|\mathrm{Jac}^{(v)}(\check{L}/\check{k})| = v^{2g} = |\mathrm{Jac}^{(v)}(\check{L}'/\check{k}')|$. Therefore the homomorphism $\phi^{(v)}_{\mathrm{jac}}$ induces an isomorphism $\check{\phi}^{(v)}\colon \mathrm{Jac}^{(v)}(\check{L}/\check{k})\ \mathrm{Jac}^{(v)}(\check{L}'/\check{k}')$. Since g is positive, the Galois groups act faithfully on the corresponding Jacobians, and by Weil's Theorem on Tate modules (see Theorem 3 on page 176 of [Mum] and Lemma 12.2 on page 122 of [Mil]) we know that, for large enough m, the Galois groups $\mathrm{Gal}(\check{L}, \check{K}) = \mathrm{Gal}(L/K)$ and $\mathrm{Gal}(\check{L}', \check{K}') = \mathrm{Gal}(L'/K')$ act faithfully on $\mathrm{Jac}^{(v)}(\check{L}/\check{k})$ and $\mathrm{Jac}^{(v)}(\check{L}'/\check{k}')$ respectively. Therefore ϕ_{gal} is injective.

Note that, in the Riemann–Hurwitz genus formula, the contribution of a nontamely ramified point is greater than the contribution in characteristic zero for the same ramification exponent. Therefore, in view of (2.1) to (2.3), by (1.4) we get the following General Theorem.

General theorem 2.4. *Recall that L/k^* is a 1-dimensional regular function field such that some member of $\mathfrak{R}(L/k^*)$ is residually rational over k^*, the genus of L/k^* is at least 2, T is a pseudogeometric DVR with quotient field k^* such that the residue field of T is perfect, E is a good projective model of L/T, $E_T = \mathfrak{R}(L/k^*) =$ the generic fiber of E over T, the DVR Θ with quotient field L is the generic point of the special fiber $\mathfrak{Z}(T_E)$ of E over T, $\phi\colon \Theta \to L' = \Theta/M(\Theta)$ is the canonical epimorphism, $k' = \phi(T)$, K is a subfield of L such that L/K is Galois and K/k^* is a simple transcendental extension, the DVR U with quotient field K is defined by putting $U = \Theta \cap K$, $K' = \phi(U)$, and by (2.3) ϕ induces an isomorphism $\phi_{\mathrm{gal}}\colon Gal(L, K) \to Gal(L', K')$.*

Now assume that all the members of $\mathfrak{R}(L/k^*)$, which are ramified over their contractions to K, are residually rational over k^*, and let $x' = \phi(x)$ with x as in (2.2). Then $K' = k'(x')$ and L'/K' is a Galois extension whose Galois group is isomorphic to the Galois group of L/K. Let $P_1, P_2, \ldots, P_a$ be the branch points of L'/K', i.e., those members of $\mathfrak{R}(K'/k')$ which are ramified in L'. Then there is a disjoint partition $\Pi_{1 \leqslant i \leqslant a}\{V_{i1}, V_{i2}, \ldots, V_{im_i}\}$ of the set of all branch points of L/K, with $m_i > 0$ and $V_{ij} \neq V_{ij'}$ for $1 \leqslant i \leqslant a$ and $1 \leqslant j < j' \leqslant m_i$, such that, for $1 \leqslant i \leqslant a$, upon letting R_i to be the unique member of D with $\phi(R_i) = P_i$ where D is as in (2.2), we have that $V_{i1}, V_{i2}, \ldots, V_{im_i}$ are exactly those branch points of L/K which belong to $\mathfrak{B}(R_i)$. Let r_i be the ramification exponent of an extension of P_i to L' and let s_i be the part of r_i prime to the residue characteristic of T. Also let r_{ij} be the ramification exponent of an extension of V_{ij} to L and let s_{ij} be the part of r_{ij} prime to the residue characteristic of T. Then for $1 \leqslant i \leqslant a$ we have that the integers $s_{i1}, s_{i2}, \ldots, s_{im_i}$ are pairwise coprime and their product equals s_i, and $r_i/s_i \geqslant r_{ij}/s_{ij}$ for $1 \leqslant j \leqslant m_i$, and moreover: $r_i = s_i \Leftrightarrow r_{ij} = s_{ij}$ for $1 \leqslant j \leqslant m_i$. Finally, if the ramification exponent of an extension to L of some member of $\mathfrak{R}(K/k^*)$ is divisible by the residue characteristic of T, then we must have $m_i > 1$ for some i.

DEFINITION 2.5.

If a good model E of L/T exists, we may express this by saying that L/k^* (or the corresponding nonsingular projective algebraic curve $\mathscr{X}$) *has a good reduction via T or modulo $M(T)$*, and we may refer to L'/k' (or the corresponding nonsingular projective algebraic curve $\mathscr{X}'$) as *a good reduction of L/k^* via T or modulo $M(T)$*; we may also call K' *the corresponding reduction of* K. In case the characteristic of T is zero and the characteristic of $T/M(T)$ is $p > 0$, briefly speaking, we may talk of *reduction modul* p instead of reduction modulo $M(T)$.*

Abhyankar's Lemma 2.6. This has many versions. For one version see pages 181–18 of [A05]. At any rate, using it we see that if k' is a field of characteristic $p > 0$ and L' is a Galois extension of $k'(x')$ which is unramified except at $x' = \infty$ and $x' = 0$ and for which the ramification exponent at $x' = 0$ is nondivisible by p, then upon letting $\Lambda = L'(k(X))$, where $X^\rho = x'$ with any positive integer ρ divisible by the ramification exponent at $x' = 0$ and k is any algebraically closed overfield of k' such that X is transcendental over k', we have that the Galois extension $\Lambda/k(X)$ is unramified except at $X = \infty$. Moreover, if the extensions of $x' = \infty$ to L' are residually rational over k then $\mathrm{Gal}(L'/K')$ remains unchanged when we replace k' by its algebraic closure and hence, in view of (3.1) of [A10], $\mathrm{Gal}(\Lambda, k(X))$ is (isomorphic to) a normal subgrou of $\mathrm{Gal}(L', k'(x'))$ and the corresponding factor group is a cyclic group whose orde divides ρ but is nondivisible by p; consequently, in view of Result 4 on page 841 o [A03], $\mathrm{Gal}(L', k'(x'))/p(\mathrm{Gal}(L', k'(x')))$ must be cyclic and we must have $\mathrm{Gal}(\Lambda, k(X)) = p(\mathrm{Gal}(L', k'(x')))$; (recall that for any finite group G, by $p(G)$ we denote the normal subgroup of G generated by all of its p-Sylow subgroups). Therefore i the situation of (2.4), if the characteristic of k' is $p > 0$ and L/K has at most three branch points, exactly one of which has ramification exponent divisible by p, then by making a suitable fractional linear transformation on x', we get a Galois extensio $L'/k'(x')$, having $x' = \infty$ as the only nontame branch point and having $x' = 0$ as the only other possible branch point, such that $\mathrm{Gal}(L'/k'(x')) = \mathrm{Gal}(L, K)$, and hence upon letting $\Lambda = L'(k(X))$, where $X^\rho = x'$ with any positive integer ρ divisible by the ramification exponent at $x' = 0$ and k is any algebraically closed overfield of k' suc that X is transcendental over k', we get a Galois extension $\Lambda/k(X)$, having $X =$ as the only branch point, such that $\mathrm{Gal}(\Lambda, k(X)) = p(\mathrm{Gal}(L, K))$.

Remark 2.7. Let $\mathscr{X}$ be a nonsingular Galois covering of the projective line $\mathscr{P}^1$ i characteristic zero. Let $r_1, r_2, \ldots, r_h$ be the ramification exponents of the branch point on $\mathscr{P}^1$. Let p be a prime number and let $s_1, s_2, \ldots, s_h$ be the prime to p parts o $r_1, r_2, \ldots, r_h$ respectively. Assume that $\mathrm{GCD}(s_i, s_j) > 1$ for all $i \neq j$. Also assume tha $r_i \equiv 0(p)$ for some i. Finally assume that the genus of $\mathscr{X}$ is at least 2. Then by (2.4) w see that $\mathscr{X}$ *cannot have a good reduction modulo p.*

3. Modular curves

Recall that for any ring A and positive integer m, the group of all m by m matrice with entries in A and with determinant *a* unit in A, is denoted by $\mathrm{GL}(m, A)$, and th subgroup of those matrices whose determinant is 1 is denoted by $\mathrm{SL}(m, A)$. Moreove $\mathrm{GL}(m, A)/\{\pm 1\}$ and $\mathrm{SL}(m, A)/\{\pm 1)$ are the factor groups by the subgroups consistin of the diagonal matrices with all the entries 1 or all the entries -1; in case of $\mathrm{SL}(m, A$ we take the latter only if $(-1)^n = 1$. Likewise $\mathrm{PGL}(m, A)$ and $\mathrm{PSL}(m, A)$ are the facto groups of $\mathrm{GL}(m, A)$ and $\mathrm{SL}(m, A)$ by their subgroups of scalar matrices, i.e., diagona matrices with all entries in the diagonal equal to each other.

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*See Remark (3.3) of § 3.

For the finite field $\mathrm{GF}(q)$ of q elements, we may write $\mathrm{GL}(m,q)$, $\mathrm{SL}(m,q),\ldots$ and so on in place of $\mathrm{GL}(m,\mathrm{GF}(q))$, $\mathrm{SL}(m,\mathrm{GF}(q)),\ldots$ and so on. Note that the *projective special linear group* $\mathrm{PSL}(m,q)$ is a finite simple group provided $m \geqslant 2$ with the exclusion of $(m,q)=(2,2)$, $(2,3)$. Likewise, the *projective special unitary group* $\mathrm{PSU}(m,q)$ is a finite simple group provide $m \geqslant 2$ with the exclusion of $(m,q)=(2,2)$, $(2,3)$, $(3,2)$. For a discussion of unitary groups, reference may be made to [Car] or [Suz]. In particular note that the *general unitary group* $\mathrm{GU}(m,q)$ is the subgroup of those members of $\mathrm{GL}(m,q^2)$ which leave a Hermitian form invariant, the *special unitary group* $\mathrm{SU}(m,q)$ is the group consisting of those members of $\mathrm{GU}(m,q)$ whose determinant is 1, and the *projective general unitary group* $\mathrm{PGU}(m,q)$ and the *projective special unitary group* $\mathrm{PSU}(m,q)$ are the factor groups of $\mathrm{GU}(m,A)$ and $\mathrm{SU}(m,A)$ by their subgroups of scalar matrices. Also recall that $|\mathrm{PGL}(2,q)|=(q+1)q(q-1)$, $|\mathrm{PGL}(2,q)/\mathrm{PSL}(2,q)|=\mathrm{GCD}(2,q+1)$, $|\mathrm{PGU}(3,q)|=(q^3+1)q^3(q^2-1)$ and $|\mathrm{PGU}(3,q)/\mathrm{PSU}(3,q)|=\mathrm{GCD}(3,q+1)$; it follows that if q is a power of the prime p then $\mathrm{PSL}(2,q)$ (resp: $\mathrm{PSU}(3,q)$) is the only quasi-p normal subgroup of $\mathrm{PGL}(2,q)$ (resp: $\mathrm{PGU}(3,q)$) such that the corresponding factor group is cyclic. Now $\mathrm{PSL}(m,q)$ and $\mathrm{PSU}(m,q)$ are two of the 16 infinite families of finite simple groups (for a survey of finite simple groups see [A10]). The third infinite family of finite simple groups relevant to us is the family of *Suzuki groups* $\mathrm{Sz}(q)$ with $q=2^{2v+1}$ where v is any positive integer. Note that $\mathrm{Sz}(q)$ is a certain subgroup of $\mathrm{GL}(4,q)$ and for its order we have $|\mathrm{Sz}(q)|=(q^2+1)q^2(q-1)$; this also holds for $q=2$, but $\mathrm{Sz}(2)$ is not simple. The relevance of the unitary groups and the Suzuki groups for us will become clear in (4.2.1) and (4.2.3) respectively.

For any positive integer n, let $\bar{\Gamma}(n)$ be the *inhomogeneous principal congruence subgroup of level* n, i.e., the kernel of the natural epimorphism

$$\mathrm{SL}(2,Z)/\{\pm 1\} \to \mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})/\{\pm 1\}.$$

Referring to [Sch] for details, $\mathrm{SL}(2,Z)/\{\pm 1\}$ acts in a natural way on the upper half plane $\mathfrak{H}$ (consisting of all complex numbers with positive imaginary part) and has the complex projective line $\mathscr{P}^1_{\mathbb{C}}$ as compactified quotient. This induces an action of $\bar{\Gamma}(n)$ on $\mathfrak{H}$ and now the compactified quotient is called the *modular curve* $\mathscr{X}(n)$ *of level* n. Clearly $\bar{\Gamma}(1)=\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})/\{\pm 1\}$ and hence $\mathscr{X}(1)=\mathscr{P}^1_{\mathbb{C}}$. Let $\mathfrak{F}_{n,\mathbb{C}}$ be the function field of $\mathscr{X}(n)$. Then $\mathfrak{F}_{1,\mathbb{C}}$ is generated over $\mathbb{C}$ by the famous transcendental function j, and $\mathfrak{F}_{n,\mathbb{C}}/\mathfrak{F}_{1,\mathbb{C}}$ is a Galois extension with Galois group $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})/\{\pm 1\}$.

Let $\mathbb{Q}_n$ be the field extension of $\mathbb{Q}$ obtained by adjoining to it all the nth roots of 1 in $\mathbb{C}$.* Then, referring to [Shi] for details, the modular curve $\mathscr{X}(n)$ is defined over $\mathbb{Q}_n$ in the sense that there are a finite number of generators $\{f_a : a \in n^{-1}\mathbb{Z}^2\backslash\mathbb{Z}^2\}$ of $\mathfrak{F}_{n,\mathbb{C}}$ over $\mathbb{C}(j)$, called Fricke functions in [Lan], which are algebraic over $\mathbb{Q}(j)$, and upon letting $\mathfrak{F}_n$ to be the field generated by them over $\mathbb{Q}(j)$ we have that the algebraic closure of $\mathbb{Q}$ in $\mathfrak{F}_n$ is $\mathbb{Q}_n$. Moreover, $\mathfrak{F}_n/\mathbb{Q}(j)$ and $\mathfrak{F}_n/\mathbb{Q}_n(j)$ are Galois extensions with Galois groups $\mathrm{GL}(2,\mathbb{Z}/n\mathbb{Z})/\{\pm 1\}$ and $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})/\{\pm 1\}$ respectively.

Henceforth assume that $n>6$. Then by the classical theory [Sch] it follows that $\mathfrak{F}_n/\mathbb{Q}_n(j)$ has exactly three branch points, which are all residually rational over $\mathbb{Q}_n$, and the ramification exponents of their extensions to $\mathfrak{F}_n$ are $2,3,n$. We can take a finite algebraic field extension $\mathbb{Q}^*_n$ of $\mathbb{Q}_{6n}$ in $\mathbb{C}$ such that, for any algebraic field extension k^*_n of $\mathbb{Q}^*_n$ in $\mathbb{C}$, upon letting K_n and L_n to be the compositums of $\mathbb{Q}_n(j)$

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*Our notation $\mathbb{Q}_n$ should not be confused with the notation $\mathbb{Q}_l$ frequently used for l-adic numbers.

and $\mathfrak{F}_n$ with k_n^* we have that the extensions to L_n of any branch point of L_n/K_n ar all residually rational over k_n^*. Now clearly K_n/k_n^* and L_n/k_n^* are 1-dimensional regula function fields, K_n/k_n^* is a simple transcendental extension, the genus of L_n/k_n^* is a least 2, L_n/K_n is a Galois extension with Galois group $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})/\{\pm 1\}$ and wit exactly three branch points, and the ramification exponents of their extensions to L are $2,3,n$. By the (Riemann–Hurwitz) genus formula, it follows that *the genus o* *L_n/k_n^* is at least* 2.

Given a prime number p, in view of Lemma 12 on page 138 of [A07] we can tak k_n^* to be the quotient field of a pseudogeometric DVR T_n such that $T_n/M(T_n)$ is a algebraic closure of $\mathrm{GF}(p)$. The following beautiful result was proved by Igusa [Igu (for a more modern treatment see 4a on page 152 of [DRa] or Chapter 2 of [MWi or Chapter 12 of [KMa]).

PROPOSITION 3.1.

If $n \not\equiv 0(p)$ then L_n/k_n^* has a good reduction L'_n/k'_n modulo $M(T_n)$.

Let K'_n be the corresponding reduction of K_n. Now, according to Feit [Fe1], i $\mathrm{GCD}(n,6)=1$ then $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})/\{\pm 1\}$ is a quasi 2-group as well as a quasi 3-grou Therefore, in view of (3.1) and Abhyankar's Lemma (2.6), by (2.4) we get the following.

Theorem 3.2. *Assume that $n \not\equiv 0(p)$ and $p=2$ or 3. Then K'_n is a simple transcendenta extension of the algebraically closed field k'_n of characteristic p, L'_n/k'_n is a 1-dimensiona regular function field of genus at least 2, L'_n/K'_n is a Galois extension with Galois grou $SL(2,\mathbb{Z}/n\mathbb{Z})/\{\pm 1\}$, and, for a suitable choice of x' with $K'_n = k'_n(x')$, we have that $x' =$ is a nontame branch point of L'_n/K'_n, $x'=0$ is the only other possible branch point o L'_n/K'_n, and the reduced ramification exponent of any extension to L of $x'=0$ is factor of $6n/p$. Moreover, if $GCD(n,6)=1$, then upon letting $\Lambda_n = L'_n(k_n(X))$, where is an element in an overfield of L'_n with $X^{6n/p}=x'$ and k_n is any algebraically close overfield of k'_n such that X is transcendental over k_n, we have that $\Lambda_n/k_n(X)$ is a Galo extension with $X=\infty$ as the only branch point and with Galois group $SL(2,\mathbb{Z}/n\mathbb{Z})/\{\pm 1\}$*

Note 3.2*. Let us call a group **anti-*p*** if it is finite and does not have any nonidentit normal subgroup whose order is a power of p; note that then a nonabelian finit simple group is always anti-p. Now in the situation of (2.3), assuming the characteristi of k' to be $p>0$, the order of the kernel of ϕ_{gal} is a power of p, and hence if $\mathrm{Gal}(L/k^*$ is an anti-p group then obviously (and hence without invoking the present proof o (2.3)) ϕ_{gal} is injective. According to Feit [Fe2], assuming $\mathrm{GCD}(n,6)=1$, the grou $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})/\{\pm 1\}$ is always anti-3, and moreover: it is anti-2⇔n is a power of single prime. Therefore in these cases of the above theorem we can disregard th present proof of (2.3).

Remark 3.3. Instead of considering good reductions when they exist, Mumford [MFo introduces the more general concept of stable reduction, where the reduction allowed to be a reducible curve but is required to be free from multiple componen and have no singularities worse than nodes; it is also required that every ration component of the reduction intersect the other components in at least 3 differe points so that the automorphism group of the reduction is always finite. He goes on to prove that it always exists provided we allow for a finite extension $\tilde{T}$ of T. In brief, start with any reduction, i.e., any arithmetic surface, over a pseudogeometric DVR T, with given generic fiber. Resolve its singularities by Abhyankar [A07] and Abhyankar [A09]. Then shrink unwanted (= exceptional) curves to get a minimal model à la [Lic] or [Saf] on which the special fiber is stable. To achieve this, we may have to make a "base change" by replacing T by an extension $\tilde{T}$ of T to a finite algebraic field extension $\tilde{k}$ of the quotient field k^* of T. So instead of saying "assume good reduction", we may say "assume the stable reduction to be good". Here the article "the" before "stable" is justified because Mumford proved that the stable reduction is essentially unique. If we follow this route, then we could shorten some of the proofs in § 1 and § 2 by citing Theorem 1.11 and Lemma 1.12 of [DMu]. We did not have to refer to [Lic] or [Saf] because we could fall back on [A01] where much of the ground work for minimal models in the arithmetical case was done.

In point of fact, Mumford was not doing all this only for reduction modulo p. His objective was to get a compact moduli variety M_g of curves of genus g. Namely, if we follow the classical approach and restrict to nonsingular curves of genus g, then the moduli variety is not compact. Permitting stable curves, makes it compact.

With the notion of stable curves at hand, we could have formulated Proposition (2.3) slightly more generally thus. For every geometrically irreducible curve of genus at least 2 over a perfect field k^*, by Corollary 2.7 of [DMu], we can find a finite algebraic field extension $\tilde{k}/k^*$ and an extension $\tilde{T}$ of T to $\tilde{k}$ such that reduction of the curve modulo $M(\tilde{T})$ is a stable curve. Given any subfield K of the function field L of the curve such that $k^* \subset K$ and L/K is Galois, by Theorem 1.11 and Lemma 1.12 of [DMu], every automorphism of L/K can be extended to a $\tilde{T}$-automorphism of the corresponding arithmetic surfaces such that no two automorphisms coincide after specialization. Therefore, we do not really need good models; it would be sufficient to demand that the stable reduction is irreducible, because then every automorphism of the special fiber defines a unique automorphism of the nonsingular model of this fiber. It seems to be a problem, however, to find a criterion, which shows irreducibility of the special fiber without proving nonsingularity.

Remark 3.4. The modular curve $\mathcal{X}(n)$ can be interpreted as parametrizing the space of all elliptic curves (and their stable reductions) with a level-n-structure, i.e., with assigned basis for the group of points whose order divides n. In Deuring [De2] and [De3], it is shown that a similar interpretation can be given to $\mathcal{X}(n)$ in characteristic $p > 0$ provided $n \not\equiv 0(p)$.* An alternative treatment of (3.2) could be based on this work of Deuring. Briefly, the inertia group of a point in the covering $\mathcal{X}(n) \to \mathfrak{R}(k_n'(j)/k_n')$ is equal to the reduced automorphism group of the elliptic curve which corresponds to that point, and hence by Deuring [De2] we obtain an explicit description of the said inertia group. According to Deuring, in characteristic 3 or 2, only the elliptic curves with $j = 0 = 12^3$ have nontrivial automorphisms. The reduced automorphism group of such a curve has order 6 or 12 according as the characteristic is 3 or 2. And so on.

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*It may be noted that, in [MWi] and [KMa], an analogous interpretation of $\mathcal{X}(n)$ for $n \equiv 0(p)$ is given, by using the so called Drinfeld bases.

4. Curves with big automorphism groups

Hurwitz Groups (4.1). By Hurwitz [Hur], a curve of genus $g \geqslant 2$ in characteristic ze has at most $84(g-1)$ automorphisms. This bound is sharp for infinitely many g. T structure of maximal automorphism groups is known: they are called Hurwitz grou and are defined as finite groups which can be generated by two elements of ord 2 and 3 respectively, whose product has order 7; they occur as Galois groups characteristic zero coverings $\mathscr{X} \rightarrow \mathscr{P}^1$ with three ramification points of exponents 2 and 7; for a survey of Hurwitz groups see [Con]. Therefore Hurwitz groups candidates for Galois groups to be realized in characteristics 2, 3, and 7.

Klein Curve 4.1.1. In § 3 we assumed $n > 6$, and so $\mathscr{X}(7)$ was the smallest level modu curve which gave rise to an unramified covering of the affine line in positi characteristic, namely $p = 2$ and $p = 3$. The genus of $\mathscr{X}(7)$ is 3 and the resulting Gal group is $\mathrm{SL}(2, \mathbb{Z}/7\mathbb{Z})/\{\pm 1\} = \mathrm{PSL}(2,7)$. In view of what was said in § 3, clearly $\mathscr{X}$ is a Hurwitz curve. Klein [Kle] gave $x^3y + y^3z + z^3x = 0$ as a homogeneous equati for a plane model of $\mathscr{X}(7)$, and showed that its full automorphism group is inde PSL(2, 7). Hence this is called the *Klein curve*. By direct calculation we see that, $p = 2$ and $p = 3$, this equation gives a nonsingular irreducible curve. This confir the fact that $\mathscr{X}(7)$ has good reductions modulo 2 and modulo 3 which, in view Abhyankar's Lemma (2.6), by (2.4)[†] *give rise to unramified coverings of the affine l with Galois group* PSL(2, 7) *for* $p = 2$ *and* $p = 3$.

Macbeath Curve 4.1.2. Macbeath [Mac] showed that the next possible genus o Hurwitz curve is 7, and constructed such a curve $\mathscr{X}$, which is now called the *Macbea curve*, with full automorphism group $G = \mathrm{PSL}(2,8)$. Regarding G as acting on projective line over GF(8), its 1-point stabilizer G_1 is the group of all affine lin transformations $x \mapsto ax + b$ with a and b in GF(8) and $a \neq 0$. Upon letting $\mathscr{X}_1$ to the quotient $\mathscr{X}/G_1$, Macbeath [Mac] has shown that $\mathscr{X}_1$ has genus zero and the Gal covering $\mathscr{X} \rightarrow \mathscr{X}_1$ has three ramification points with exponents 2, 7, and 7. We can fi a specialization of $\mathscr{P}^1$ modulo 3 such that no two of these three points coincide, a then, in characteristic 3, construct a nonsingular covering with the same ramificati exponents which is a specialization of the covering from characteristic zero. Since three ramification points are tame, this covering again has genus 7. Thus $\mathscr{X}$ has go reduction modulo 3 and hence, in view of Abhyankar's Lemma (2.6), by (2.4)** *get an unramified covering of the affine line in characteristic* 3 *with Galois gr* PSL(2, 8). As indicated by Serre [Se6], an explicit equation for this covering can deduced from the discussion of the Macbeath curve given in Serre [Se4].

Positive Characteristic 4.2. For an algebraically closed field k of positive characteris p, it is no longer true that a curve of genus $g \geqslant 2$ has at most $84(g-1)$ automorphis In an effort to get some other bound, Leopoldt [Leo], Stichtenoth [Sti] and He [Hen] have studied curves with big automorphism groups G; in Satz 1 of his pap Henn gives a complete list of all curves for which $|G| \geqslant 8g^3$. In all cases, the cur are coverings of the projective line $\mathscr{P}_k^1$ with exactly one wild and at most one tame ramification point; (it may be noted that by pages 536–537 of [Sti], this is so whenever $g > 2$ and $|G| > 24g^2$). Using Abhyankar's Lemma (2.6), we can get rid of the tame ramification point, if it occurs, and therefore all these curves lead to examples of unramified coverings of $\mathscr{A}_k^1$. There are the following four series, all of which are given by equations for their affine plane models, where ν is any positive integer and where "can be realized" means "belongs to $\pi_A(\mathscr{A}_k^1)$".

[†] Since PSL(2, 7) is simple, the present proof of (2.3) can be avoided; see Note (3.2*).

**Since PSL(2, 8) is simple, the present proof of (2.3) can be avoided; see Note (3.2*).

4.2.1. The unitary Fermat curve $x^n + y^n + 1 = 0$ with $n = p^\nu + 1$. Here $G = \mathrm{PGU}(3, p^\nu)$, and hence $\mathrm{PSU}(3, p^\nu)$ can be realized in characteristic p. (Henn [Hen] writes $\mathrm{PGU}(3, p^{2\nu})$ and $\mathrm{PSU}(3, p^{2\nu})$ for what we have called $\mathrm{PGU}(3, p^\nu)$ and $\mathrm{PSU}(3, p^\nu)$ respectively). (Another way of realizing $\mathrm{PSU}(3, p^\nu)$, by using Deligne–Lusztig curves, is indicated in Serre [Se3]).

4.2.2. The Fermat-like curve $x^n + y^m + 1 = 0$ with $n = p^\nu + 1$ and m a proper divisor of n. In this case, G has $C = \mathbb{Z}/m\mathbb{Z}$ as its centre, and $G/C \cong \mathrm{PGL}(2, p^\nu)$. Therefore $\mathrm{PSL}(2, p^\nu)$ can be realized in characteristic p. (For other ways of realizing $\mathrm{PSL}(2, p^\nu)$ see [A10] and [A16]).

4.2.3. The equation $x^q + x = y^{q+q'} + y^{q'+1}$ with $q = 2q'^2$ and $q' = 2^\nu$, in characteristic $p = 2$ has the Suzuki group $\mathrm{Sz}(2^{2\nu+1})$ as automorphism group. Therefore $\mathrm{Sz}(2^{2\nu+1})$ can be realized in characteristic $p = 2$. (Another way of realizing $\mathrm{Sz}(2^{2\nu+1})$, by using Deligne–Lusztig curves, is indicated in Serre [Se3]).

4.2.4. The equation $x^p - x = y^{1+p^n}$ for $n > 1$ also defines a curve with a big automorphism group G in characteristic p. The corresponding curve is a covering of $\mathscr{P}_k^1$ with only one ramification point, so G is explicitly realized; the structure of G has been determined by Serre [Se6].

5. Use of unramified abelian coverings

If $\mathscr{Y} \to \mathscr{A}_k^1$ is an unramified covering of the affine line over an algebraically closed field k of positive characteristic p, and if $\mathscr{Y}^* \to \mathscr{Y}$ is an unramified covering of $\mathscr{Y}$, then, of course, we get as new unramified covering $\mathscr{Y} \to \mathscr{A}_k^1$. This holds in particular, even if we have an unramified covering $\mathscr{X}^* \to \mathscr{X}$ where $\mathscr{X} \to \mathscr{P}_k^1$ is the corresponding covering of the projective line which is ramified only at ∞. Among the unramified coverings $\mathscr{X}^* \to \mathscr{X}$, the abelian ones are well-known: They correspond biuniquely to the finite subgroups of the Jacobian $\mathrm{Jac}(\mathscr{X})$ (see for example [De1] or [Se1]). Therefore we get the following.

Theorem 5.1. *Let $\mathscr{X} \to \mathscr{P}_k^1$ be a Galois covering with Galois group G, ramified only at ∞, and let N be a finite subgroup of Jac $(\mathscr{X})$, invariant under all automorphisms of $\mathscr{X}$. Then there exists a covering $X^* \to \mathscr{P}_k^1$, ramified only at ∞, whose Galois group G^* is an extension of G with kernel N.*

This implies, in particular, that G^* is a quasi p-group. Note that for group extensions which are *known* to be quasi-p, Serre [Se2] has proved a stronger result which says that: with G as above, for any extension

$$0 \rightarrow N \rightarrow G^* \rightarrow G \rightarrow 0$$

with solvable kernel N and a quasi-p group G, we have $G^* \in \pi_A(\mathscr{A}_k^1)$.

By corollary 1 on page 127 of [Wei] we know that, for any positive integer l nondivisible by p, the points of Jac($\mathscr{X}$) whose order divides l constitute a subgroup which is isomorphic to $(\mathbb{Z}/l\mathbb{Z})^{2g}$. Therefore as a consequence of (5.1) we get the following.

COROLLARY 5.2

Let $\mathscr{X} \rightarrow \mathscr{P}_k^1$ be a Galois covering with Galois group G, ramified only at ∞, and let g be the genus of $\mathscr{X}$. Then, for any l prime to p, there exists an extension

$$0 \rightarrow (\mathbb{Z}/l\mathbb{Z})^{2g} \rightarrow G^* \rightarrow G \rightarrow 0$$

for which we have $G^* \in \pi_A(\mathscr{A}_k^1)$.

As in §4 of [A12], while keeping G unchanged, we can make the genus of $\mathscr{X}$ arbitrarily large. Therefore by (5.2) we get the following.

COROLLARY 5.3

Given any $1 \neq G \in \pi_A(\mathscr{A}_k^1)$, for every positive integer $l \not\equiv 0(p)$ and for infinitely many positive integers γ, some extension of G with kernel $(\mathbb{Z}/l\mathbb{Z})^{2\gamma}$ belongs to $\pi_A(\mathscr{A}_k^1)$.

6. Problems

Problem 6.1. Given an algebraically closed field k of characteristic $p > 0$, for several quasi p-groups G we have constructed Galois extensions $\Lambda/k(X)$ which are ramified only at $X = \infty$ and whose Galois group is G. It would be interesting to find explicit expressions for small degree monic polynomials in Y with coefficients in $k[X]$ and with splitting field Λ.

Problem 6.2. In (2.7) we have given necessary conditions for the existence of good reduction. Can we refine these into sufficient conditions?

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A footnote to mean square value of Dirichlet L-series

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Abstract. Following appropriate use of approximate functional equation for Hurwitz Zeta function, we obtain upper bounds for

$$S(\sigma+it)=\sum_{\chi(\operatorname{mod} q)}|L(\sigma+it,\chi)|^2 \text{ for } 0<\sigma\leqslant\tfrac{1}{2} \text{ and } t\geqslant 3.$$

Here for $s=\sigma+it$, $L(s,\chi)$ denotes Dirichlet L-series for character $\chi(\operatorname{mod} q)$. In particular, we obtain $S(\frac{1}{2}+it)\ll q\log qt+t^{5/8}q^{-1/8}$, which is an improvement in the range $q<|t|<q^{11/7}$, on hitherto best known result. This incidentally gives $S(\frac{1}{2}+it)\ll q\log 3q$ for $|t|<q^{9/5}$.

Keywords. Dirichlet L-series

1. Introduction

For integer $q\geqslant 1$, let $\chi(\operatorname{mod} q)$ be a Dirichlet character modulo q and let $L(s,\chi)$ be the corresponding Dirichlet L-series, where $s=\sigma+it$. For $0<\alpha\leqslant 1$, let $\zeta(s,\alpha)$ be Hurwitz Zeta function and let $\zeta_1(s,\alpha)=\zeta(s,\alpha)-\alpha^{-s}$ and let $S(s)=\sum_{\chi(\operatorname{mod} q)}|L(\sigma+it,\chi)|^2$, where $s=\sigma+it$, σ,t being real.

Notation. In what follows, $\phi(n)$ shall denote Euler's function. $\varepsilon>0$ will be arbitrary. The 0 and $\ll$ constants depend on σ only.

The object of this note is to get an upper bound for $S(s)=\sum_{\chi(\operatorname{mod} q)}|L(\sigma+it,\chi)|^2$ for $0<\sigma\leqslant\frac{1}{2}$. In particular for $q<|t|<q^{11/7}$, our result is an improvement over the hitherto best known upper bound for $S(\frac{1}{2}+it)$. Gallagher [1] has shown that $S(\frac{1}{2}+it)\ll(q+|t|)\log q(|t|+2)$. This gives $S(\frac{1}{2}+it)\ll q\log(q+2)$ for $|t|\leqslant q$.

By Lindelöf's hypothesis, note that $S(\frac{1}{2}+it)\ll q^{1+\varepsilon}\cdot t^{\varepsilon}$, uniformly in q and t. By generalizing Atkinson's formula (for Riemann Zeta function) to $\int_0^T S(\frac{1}{2}+it)\mathrm{d}t$, as a corollary, Meurman [2] has shown that $S(\frac{1}{2}+it)\ll(q+(qt)^{1/3})(qt)^{\varepsilon}$. This is the best upper bound hitherto known, for $S(\frac{1}{2}+it)$. This gives $S(\frac{1}{2}+it)\ll q^{1+\varepsilon}$ for $|t|<q^2$.

By appropriate use of approximate functional equation for Hurwitz Zeta function [3] and by using the estimate for $\int_0^1|\zeta_1(\sigma+it,\alpha)|^2\mathrm{d}\alpha$ ([4]) in critical strip, we obtain almost immediately the following.

Theorem. *For $q \geqslant 1$ and $t \geqslant 3$; $0 < \sigma \leqslant \frac{1}{2}$, we have*

$$\text{I)} \quad \sum_{\chi(\operatorname{mod} q)} |L(\sigma + it, \chi)|^2 \ll q^{2-2\sigma} t^{1-2\sigma} \cdot \log qt + q^{1-2\sigma} \cdot t^{3/2-\sigma} (\log t)^{1/2}$$

and

$$\ll (q^{2-2\sigma} \cdot t^{1-2\sigma} + q^{1-\sigma} \cdot t^{1-\sigma}) \log t, \quad \textit{for } t \geqslant q;$$

the log factor occurring for $\sigma = \frac{1}{2}$ only.

$$\text{II)} \quad \sum_{\chi(\operatorname{mod} q)} |L(\tfrac{1}{2} + it, \chi)|^2 \ll q \log qt + t^{5/8} q^{-1/8} \cdot \log qt.$$

COROLLARY

We have $\sum_{\chi(\operatorname{mod} q)} |L(\frac{1}{2} + it, \chi)|^2 \ll q \log(q+2)$ for $|t| \ll q^{9/5}$.

Note. Our estimate II) for $\sum_{\chi(\operatorname{mod} q)} |L(\frac{1}{2} + it, \chi)^2|$ is better than Meurman's [2] for $q < |t| < q^{11/7}$. A corresponding result I) for $\frac{1}{2} < \sigma < 1$ also follows immediately. We shall need the following lemmas.

Lemma 1. *Let $h > 0$ be a constant. Let*

$$0 < \sigma < 1, \quad 0 < \alpha \leqslant 1; \quad x > h > 0, \quad y > h > 0; \quad 2\pi xy = t$$

and

$$\psi(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s).$$

Then

$$\zeta(s, \alpha) = \sum_{0 \leqslant n \leqslant x} (n + \alpha)^{-s} + \psi(s) \sum_{1 \leqslant n \leqslant y} \exp(-2\pi i n \alpha) \cdot n^{s-1} + O(x^{-\sigma} \log(y+2)) + O(t^{1/2-\sigma} \cdot y^{\sigma-1}).$$

Note. For proof of lemma 1, see Rane [3]. Note, $\log(y+2)$ can be dropped.

Lemma 2. *Let $f(x)$ be a real function with a continuous decreasing derivative $f'(x)$ in (a, b) and let $f'(b) = \alpha$, $f'(a) = \beta$. Then*

$$\sum_{a < n \leqslant b} \exp(2\pi i f(n)) = \sum_{\alpha - \eta < \nu < \beta + \eta} \int_a^b \exp(2\pi i (f(x) - \nu x)) \mathrm{d}x + O(\log(\beta - \alpha + 2)),$$

where $0 < \eta < 1$ is any real number.

Note. Our lemma 2 is lemma 4.10 of Titchmarsh [5].

Lemma 3. *Let $F(x)$ be a real function, twice differentiable and let $F''(x) \geqslant r > 0$* or

$F''(x) \leqslant -r < 0$ on interval (a,b). Then

$$\int_a^b \exp[iF(x)]\,\mathrm{d}x \ll \frac{1}{\sqrt{r}}.$$

Note. Our lemma 3 is lemma 4.5 of Titchmarsh [5].

Proof of Theorem. For $s = \sigma + it$, with $0 < \sigma \leqslant \frac{1}{2}$ and $t \geqslant 3$, we have

$$\begin{aligned} S(s) &= \sum_{\chi(\mathrm{mod}\, q)} |L(\sigma + it, \chi)|^2 \\ &= \frac{\phi(q)}{q^{2\sigma}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left|\zeta\left(s, \frac{a}{q}\right)\right|^2 \ll \frac{\phi(q)}{q^{2\sigma}} \sum_{a=1}^{q} \left|\zeta\left(s, \frac{a}{q}\right)\right|^2 \\ &\ll \frac{\phi(q)}{q^{2\sigma}} \sum_{a=1}^{q} \left|\frac{q^\sigma}{a^\sigma} + \zeta_1\left(s, \frac{a}{q}\right)\right|^2 \ll \phi(q) \sum_{a=1}^{q} a^{-2\sigma} + \frac{\phi(q)}{q^{2\sigma}} \sum_{a=1}^{q} \left|\zeta_1\left(s, \frac{a}{q}\right)\right|^2. \end{aligned}$$

Next, by Euler's summation formula,

$$\begin{aligned} \sum_{a=1}^{q} \left|\zeta_1\left(s, \frac{a}{q}\right)\right|^2 &= \int_0^q \left|\zeta_1\left(s, \frac{a}{q}\right)\right|^2 \mathrm{d}a \\ &\quad + \int_0^q \mathrm{d}a\left(a - [a] - \tfrac{1}{2}\right)\frac{\mathrm{d}}{\mathrm{d}a}\left(\zeta_1\left(s, \frac{a}{q}\right)\zeta_1\left(\bar{s}, \frac{a}{q}\right)\right) + \frac{(|\zeta_1(s,1)|^2 - |\zeta_1(s,0)|^2)}{2}. \end{aligned}$$

Note $|\zeta_1(s,1)|^2 - |\zeta_1(s,0)|^2 = |\zeta(s) - 1|^2 - |\zeta(s)|^2 \ll |\zeta(s)| \ll t^{1/2-(1/2)\sigma}$, where $\zeta(s)$ is Riemann Zeta function. Also

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}a}\left(\zeta_1\left(s, \frac{a}{q}\right)\zeta_1\left(\bar{s}, \frac{a}{q}\right)\right) &= \frac{-\bar{s}}{q}\zeta_1\left(s, \frac{a}{q}\right)\zeta_1\left(\bar{s}+1, \frac{a}{q}\right) \\ &\qquad + \text{(its complex conjugate)} \\ &\ll \frac{|s|}{q}\left|\zeta_1\left(s, \frac{a}{q}\right)\zeta_1\left(s+1, \frac{a}{q}\right)\right|. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{a=1}^{q} \left|\zeta_1\left(s, \frac{a}{q}\right)\right|^2 &= \int_0^q \left|(\zeta_1\left(s, \frac{a}{q}\right)\right|^2 \mathrm{d}a \\ &\quad + O\left(\frac{|s|}{q}\int_0^q \left|\zeta_1\left(s, \frac{a}{q}\right)\zeta_1\left(\bar{s}+1, \frac{a}{q}\right)\right| \mathrm{d}a + O(t^{1/2-\sigma/2})\right) \\ &= q\int_0^1 |\zeta_1(s,\alpha)|^2 \mathrm{d}\alpha + O\left(|t|\int_0^1 \mathrm{d}\alpha |\zeta_1(s,\alpha)\zeta_1(s+1,\alpha)|\right) + O(t^{1/2-\sigma/2}). \end{aligned}$$

On using Hölder's inequality and noting $|\zeta_1(s+1,\alpha)| \ll 1$ for $\sigma > 0$, we have,

$$\begin{aligned} \sum_{a=1}^{q} \left|\zeta_1\left(s, \frac{a}{q}\right)\right|^2 &= q\int_0^1 |\zeta_1(s,\alpha)|^2 \mathrm{d}\alpha \\ &\quad + O\left(|t|\left(\int_0^1 |\zeta_1(s,\alpha)|^2 \mathrm{d}\alpha\right)^{1/2}\right) + O(t^{1/2-\sigma/2}). \end{aligned}$$

From [4], we have

$$\int_0^1 |\zeta_1(s,\alpha)|^2 \mathrm{d}\alpha = |\psi^2(s)| \sum_{n \leqslant t/2\pi} n^{2\sigma-2} + O(1) \ll t^{1-2\sigma} \cdot \log t,$$

for $0 < \sigma \leqslant \frac{1}{2}$, the log factor occurring for $\sigma = \frac{1}{2}$ only. Thus,

$$\sum_{a=1}^{q} \left| \zeta_1\left(s, \frac{a}{q}\right) \right|^2 \ll qt^{1-2\sigma} \cdot \log t + t^{3/2-\sigma} (\log t)^{1/2}$$

$$+ t^{1/2-\sigma/2} \ll qt^{1-2\sigma} \cdot \log t + t^{3/2-\sigma} \cdot (\log t)^{1/2}.$$

Thus

$$S(s) \ll \phi(q) \sum_{a=1}^{q} a^{-2\sigma} + \frac{\phi(q)}{q^{2\sigma}} (qt^{1-2\sigma} \log \mathrm{t} + t^{3/2-\sigma} (\log t)^{1/2})$$

$$\ll q^{2-2\sigma} \log 3q + q^{2-2\sigma} \cdot t^{1-2\sigma} \log t + q^{1-2\sigma} t^{3/2-\sigma} (\log t)^{1/2}$$

$$\ll q^{2-2\sigma} t^{1-2\sigma} \log qt + q^{1-2\sigma} t^{3/2-\sigma} \cdot (\log t)^{1/2}$$

$$\ll q^{2-2\sigma} t^{1-2\sigma} \log qt \text{ for } t \leqslant q.$$

Next, we shall prove the result for $t \geqslant q$. Let $t \geqslant q$. Then, by lemma 1, we have for $x = c(t/q)^{1/2}$ (and $y = c(qt)^{1/2}$) where

$$c = \frac{1}{(2\pi)^{1/2}}, \quad \zeta\left(s, \frac{a}{q}\right) = \sum_{0 \leqslant n \leqslant c(t/q)^{1/2}} \left(n + \frac{a}{q}\right)^{-s}$$

$$+ \psi(s) \sum_{n \leqslant c(qt)^{1/2}} \exp(-2\pi i n a/q) \cdot n^{s-1} + O(1).$$

Noting $|\psi(\sigma + it)| \ll t^{1/2-\sigma}$, we have

$$\sum_{a=1}^{q} \left| \zeta\left(s, \frac{a}{q}\right) \right|^2 \ll \sum_{a=1}^{q} \left| \sum_{0 \leqslant n \leqslant c(t/q)^{1/2}} \left(n + \frac{a}{q}\right)^{-s} \right|^2$$

$$+ t^{1-2\sigma} \cdot \sum_{a=1}^{q} \left| \sum_{n \leqslant c(qt)^{1/2}} \exp(-2\pi i n a/q) \cdot n^{s-1} \right|^2 + q.$$

Now

$$\sum_{a=1}^{q} \left| \sum_{n \leqslant c(qt)^{1/2}} \exp(-2\pi i n a/q) \cdot n^{s-1} \right|^2$$

$$= \sum_{m,n \leqslant c(qt)^{1/2}} \sum m^{s-1} n^{\bar{s}-1} \left(\sum_{a=1}^{q} \exp(2\pi i a/q)(n-m) \right)$$

$$= q \sum_{\substack{m,n \leqslant c(qt)^{1/2} \\ m \equiv n (\mathrm{mod}\, q)}} \sum m^{s-1} \cdot n^{\bar{s}-1} = q^{2\sigma-1} \cdot \sum_{a=1}^{q} \left| \sum_{0 \leqslant n \leqslant c(t/q)^{1/2} - (a/q)} \left(n + \frac{a}{q}\right)^{s-1} \right|^2$$

$$\ll q^{2\sigma-1} \cdot \sum_{a=1}^{q} \left| \sum_{1 \leqslant n \leqslant c(t/q)^{1/2} - (a/q)} \left(n + \frac{a}{q}\right)^{s-1} \right|^2 + q \sum_{a=1}^{q} a^{2\sigma-2}$$

$\ll q^{\sigma} \cdot t^{\sigma} + q \cdot \log 3q$, by taking trivial bounds.

Thus,

$$S(s) \ll q^{1-2\sigma} \sum_{a=1}^{q} \left|\zeta\left(s, \frac{a}{q}\right)\right|^2 \ll q^{1-2\sigma} \left\{ q \log qt \right.$$

$$+ \sum_{a=1}^{q} q^{2\sigma} a^{-2\sigma} + \sum_{a=1}^{q} \left| \sum_{1 \leqslant n \ll (t/q)^{1/2}} \left(n + \frac{a}{q}\right)^{-s} \right|^2$$

$$\left. + t^{1-2\sigma} \left(q \cdot \sum_{a=1}^{q} a^{2\sigma-2} + q^{2\sigma-1} \cdot \sum_{a=1}^{q} \left| \sum_{1 \leqslant n \leqslant (t/q)^{1/2} - (a/q)} \left(n + \frac{a}{q}\right)^{s-1} \right|^2 \right) \right\}$$

$$\ll q^{2-2\sigma} \cdot t^{1-2\sigma} \cdot \log qt + q^{1-\sigma} \cdot t^{1-\sigma}, \text{ by taking trivial bounds,}$$

the log factor occurring for $\sigma = \frac{1}{2}$ only.

Next, we shall obtain another estimate for $S(\frac{1}{2} + it)$ for $t \geqslant q$. As before,

$$S(\tfrac{1}{2} + it) \leqslant \sum_{a=1}^{q} |\zeta(\tfrac{1}{2} + it, \frac{a}{q})|^2$$

$$\ll q + \sum_{a=1}^{q} \left| \sum_{0 \leqslant n \leqslant c(t/q)^{1/2}} \left(n + \frac{a}{q}\right)^{-1/2 - it} \right|^2$$

$$+ \sum_{a=1}^{q} \left| \sum_{n \leqslant c(qt)^{1/2}} \exp(-2\pi i n a/q) \cdot n^{-1/2 + it} \right|^2$$

$$\ll \sum_{a=1}^{q} \left| \sum_{0 \leqslant n \leqslant c(t/q)^{1/2}} \left(n + \frac{a}{q}\right)^{-1/2 - it} \right|^2 + q$$

$$\ll \sum_{a=1}^{q} \frac{q}{a} + \sum_{a=1}^{q} \left| \sum_{1 \leqslant n \leqslant c(t/q)^{1/2}} (n + (a/q))^{-1/2 - it} \right|^2 + q \log qt$$

$$\ll q \log qt + \sum_{a=1}^{q} \sum_{0 \leqslant n \leqslant c(t/q)^{1/2}} \frac{1}{(n + (a/q))}$$

$$+ \sum_{a=1}^{q} \sum_{1 \leqslant m < n \leqslant c(t/q)^{1/2}} (m + (a/q))^{-1/2 + it} (n + (a/q))^{-1/2 - it}$$

$$\ll q \log qt + \sum_{1 \leqslant m < n \leqslant c(t/q)^{1/2}} \left(\sum_{a=1}^{q} (m + (a/q))^{-1/2 + it} (n + (a/q))^{-1/2 - it} \right).$$

Now, we estimate $A(m,n) = \sum_{a=1}^{q} (m + (a/q))^{-1/2 + it} (n + (a/q))^{-1/2 - it}$ for $m \neq n$.

Now,

$$A(m,n) = \sum_{a=1}^{q} (m + (a/q))^{-1/2} (n + (a/q))^{-1/2}$$

$$\times \exp\left[2\pi i \left(\frac{t}{2\pi} \left(\log(mq + a) - \log(nq + a) \right) \right) \right]$$

$$\ll m^{-1/2} n^{-1/2} \sum_{a=1}^{q_0} \exp\left[2\pi i \left(\frac{t}{2\pi} \left(\log(mq + a) - \log(nq + a) \right) \right) \right]$$

for some integer $q_0 = q_0(m, n)$ with $1 \leqslant q_0 \leqslant q$, where q_0 depends on m, n.

Next, we apply lemma 2 to estimate

$$\sum_{a=1}^{q_0} \exp\left[2\pi i\left(\frac{t}{2\pi}\left(\log(mq+a)-\log(nq+a)\right)\right)\right]$$

Here

$$f(a)=\frac{t}{2\pi}\cdot(\log(mq+a)-\log(nq+a)),$$

$$f'(a)=\frac{t}{2\pi}\cdot\frac{(n-m)q}{(mq+a)(nq+a)},$$

$$f''(a)=\frac{t}{2\pi}\cdot\frac{(m^2-n^2)q^2+2(m-n)qa}{(mq+a)^2(nq+a)^2}$$

Thus,

$$-f''(a)\gg\frac{t}{q^2}\cdot\frac{(n^2-m^2)}{m^2n^2}\gg\frac{t}{q^2}\cdot\frac{(n-m)}{m^2n}.$$

Thus by lemma 2,

$$\sum_{a=1}^{q_0}\left(m+\frac{a}{q}\right)^{it}\left(n+\frac{a}{q}\right)^{-it}$$

$$=O(\log qt)+\sum_{f'(q_0)-\eta<v<f'(0)+\eta}\int_0^{q_0}\left(m+\frac{a}{q}\right)^{it}\left(n+\frac{a}{q}\right)^{-it}$$

$$\times\exp(-2\pi iva)\mathrm{d}a.$$

Now, the number of terms in $\sum_{v}$

$$\leqslant(f'(O)-f'(q_0))+1\leqslant(f'(O)-f'(q))+1\ll\frac{t}{q}\frac{(n-m)}{m^2n}+1.$$

Next, by lemma 3,

$$\int_0^{q_0}\left(m+\frac{a}{q}\right)^{it}\left(n+\frac{a}{q}\right)^{-it}\cdot\exp(-2\pi iva)\mathrm{d}a\ll\frac{q}{\sqrt{t}}\frac{m\sqrt{n}}{\sqrt{n-m}}.$$

Thus

$$\sum_{1\leqslant m<n\leqslant c(t/q)^{1/2}}\sum A(m,n)$$

$$\ll\sum_{1\leqslant m<n\leqslant c(t/q)^{1/2}}\sum m^{-1/2}n^{-1/2}\left\{\left(\frac{t}{q}\cdot\frac{(n-m)}{m^2n}+1\right)\left(\frac{q}{\sqrt{t}}\cdot\frac{m\sqrt{n}}{\sqrt{n-m}}\right)+O(\log qt)\right\}$$

$$\ll\sum_{1\leqslant m<n\ll(t/q)^{1/2}}\sum m^{-1/2}n^{-1/2}\left(\frac{t}{q}\frac{(n-m)}{m^2n}\right)\left(\frac{q}{\sqrt{t}}\frac{m\sqrt{n}}{\sqrt{n-m}}\right)$$

$$+\frac{q}{\sqrt{t}}\sum_{m<n\ll(t/q)^{1/2}}\sum\frac{\sqrt{m}}{\sqrt{n-m}}+(t/q)^{1/2}\log qt.$$

$$\ll \sqrt{t}\sum_{1\leqslant m<n\ll(t/q)^{1/2}}\sum m^{-3/2}.\frac{\sqrt{n-m}}{n}+\sqrt{t}+(t/q)^{1/2}\cdot\log qt.$$

$$\ll \sqrt{t}\sum_{1\leqslant m<n\ll(t/q)^{1/2}}\sum n^{-1/2}m^{-3/2}+\sqrt{t}\log qt$$

$$\ll \sqrt{t}\sum_{1\leqslant m<n\leqslant(t/q)^{1/4}}\sum m^{-3/2}n^{-1/2}+\sqrt{t}\cdot\sum_{(t/q)^{1/4}<m<n\leqslant(t/q)^{1/2}}\sum m^{-3/2}n^{-1/2}$$

$$+\sqrt{t}\log qt.$$

$$\ll \sqrt{t}\sum_{n\leqslant(t/q)^{1/4}} n^{-1/2}\sum_{m\leqslant n} m^{-3/2}+\sqrt{t}\sum_{n\leqslant(t/q)^{1/2}} n^{-1/2}\left(\sum_{(t/q)^{1/4}<m<n} m^{-3/2}\right)$$

$$+\sqrt{t}\log qt$$

$$\ll \sqrt{t}\sum_{n\leqslant(t/q)^{1/4}} n^{-1/2}+\sqrt{t}\sum_{n\leqslant(t/q)1/2} n^{-1/2}.\left(\frac{t}{q}\right)^{-1/8}\cdot\left(\sum_{m\leqslant n} m^{-1}\right)$$

$$+\sqrt{t}\log qt$$

$$\ll \sqrt{t}\left(\left(\frac{t}{q}\right)^{1/8}+\left(\frac{t}{q}\right)^{1/8}\log qt\right)+\sqrt{t}\log qt\ll t^{5/8}q^{-1/8}\log qt.$$

Thus,

$$S(\tfrac{1}{2}+it)\ll q\log qt+t^{5/8}q^{-1/8}\cdot\log qt.$$

This is so when $t\geqslant q$. For $t\leqslant q$, we have already shown using Euler's summation formula that $S(\frac{1}{2}+it)\ll q\log(q+2)$.

This completes the proof of our theorem.

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Matrix summability of a factored Fourier series

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Abstract. This paper reports a result for proving a triangular matrix summability of a factored Fourier series by extending the theorem on Nörlund summability of a factored Fourier series at $t = x$ when $\phi(t) \in B \cdot V$ in $(0, \pi)$ due to Singh [4] (*Indian J. Math.* **9** 227–236). The result generalizes the theorem of Varshney [5] (*Proc. Am. Math. Soc.* **10**, 784–789) and that of Singh.

Keywords. Matrix summability; factored Fourier series; triangular matrix.

1. Introduction

Let Σu_n be an infinite series and $\{s_n\}$ the sequence of its partial-sums. Let $T \equiv (a_{m,n})$ be an infinite matrix with real or complex elements. Let us consider

$$T_m = \sum_{n=1}^{\infty} a_{m,n} s_n. \tag{1}$$

and we assume that the right hand side series is convergent for all $m = 0, 1, 2, \ldots$. Then $\{T_m\}$ represents the T-transform of $\{s_n\}$ generated by the matrix $T \equiv (a_{m,n})$. If $T_m \to s$, as $m \to \infty$, we say that the series Σu_n is summable by $T \equiv (a_{m,n})$, or simply T-summables to s. The series Σu_n is said to be absolutely summable if $\{t_n\} \in B.V$.

The matrix

$$a_{m,n} = \begin{cases} A_{m-n}^{\alpha-1}, & 0 \leqslant n \leqslant m, \\ 0, & n > m \end{cases} \tag{2}$$

where A_m^{α}'s are Cesàro numbers given by

$$(1 - x)^{-\alpha-1} = \sum_{k=0}^{\infty} A_k^{\alpha} x^k, \quad |x| < 1 \tag{3}$$

defines the so called Cesàro method (C, α), $\alpha > -1$. The absolute summability (C, α) is denoted by $|C, \alpha|$.

2.

Let $f(t)$ be a periodic function with period 2π and integrable-L over $(-\pi, \pi)$. W
assume without loss of generality that the constant term in the Fourier series of $f(t$
is zero, so that

$$f(t) = \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \Sigma A_n(t) \quad (4$$

and

$$\int_{-\pi}^{\pi} f(t) \mathrm{d}t = 0.$$

We use the following notations

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\},$$

$$\Delta_k a_{n,k} = a_{n,k} - a_{n,k+1},$$

$$\nabla_k a_{n,k} = a_{n,k} - a_{n,k-1}.$$

Also, $[x]$ denotes the integrals part of x.

$$g(n,t) = \frac{2}{\pi} \sum_{k=1}^{n} a_{n,k} k a_{k,0} \sin kt.$$

3.

In 1959, Varshney [5] showed that if $\phi(t) \in B.V(0,\pi)$, then the factored Fourier serie
$\Sigma(A_n(t)/\log n)$ is summable $|N,(1/n+1)|$ at $t = x$. This theorem was set in a mo
general form by Singh [4], who proved the following theorem.

Theorem 1. *If* $\phi(t) \in B.V(0,\pi)$, *then the factored Fourier series* $\Sigma[(n+1)p_n/P_n]\ A_n($
is summable $|N, p_n|$ *at* $t = x$, *where* $\{p_n\}$ *is a positive non-increasing convex sequenc*
such that $\{(n+1)p_n/P_n\} \in B.V.$

We extend the scope of the above theorem by considering the absolute triangul
matrix summability of the factored Fourier series. We establish

Theorem. *Let* $T = (a_{n,k})$ *be an absolute regular triangular matrix. Let* $\sum_{r=k}^{n} a_{n,r} = A_{n}$
and $A_{n,0} = 1$ *for all* $n \geqslant 0$. *If* $\phi(t) \in B \cdot V.\ (0,\pi)$, *the factored Fourier series* $\Sigma n a_{n,0} A_n($
is summable $|T|$, *at* $t = x$, *provided that for every* k,

(i) $a_{n-1,k} \geqslant a_{n,k+1} \geqslant a_{n,k}$ *for all* n, k *with* $k \leqslant n-1$;
(ii) $(n+1)(a_{n-1,k} - a_{n,k+1}) \geqslant \Delta_k (n-k) a_{n,k}$;

(iii) $\Sigma \frac{a_{n,0}}{n+1} < \infty$

and

(iv) $\sum_{n=k}^{\infty} \frac{A_{n,n-k}}{n+1} < \infty$.

4.

We require the following lemmas for the proof of our theorem.

Lemma 1. ([2], *Lemma* 5.11) *If* $\{p_n\}$ *is a non-negative and non-increasing sequence, then for* $0 \leqslant a < b < \infty$, $0 \leqslant t \leqslant \pi$ *and for every n*

$$\left| \sum_{k=a}^{b} p_k \exp[i(n-k)t] \right| \leqslant A\, P_\tau$$

where A is an absolute constant and $\tau = [\frac{1}{t}]$.

Proceeding as above, Kishore and Hotta [3] proved the following result for triangular matrix. *If* $\{a_{n,k}\}_{k=0}^{n}$ *is a non-negative and non-decreasing sequence with respect to k, then for* $0 \leqslant a < b < \infty$, $0 \leqslant t \leqslant \pi$ *and for every n*,

$$\left| \sum_{k=a}^{b} a_{n,n-k} \exp[i(n-k)t] \right| \leqslant B\, A_{n,n-\tau}$$

where B is an absolute constant and $\tau = [1/t]$.

Lemma 2. [1] *Let conditions* (i) *to* (iii) *of the theorem hold. If the sequence of partial sums of* Σu_n *be bounded, then a necessary and sufficient condition for it to be summable* $|T|$ *is that*

$$\sum_{n=1}^{\infty} \frac{1}{n+1} \left| \sum_{k=0}^{\infty} a_{n,k}(k+1)u_k \right| < \infty.$$

Lemma 3. *If* $a_{n-1,k} \geqslant a_{n,k}$ *for all n, k such that* $k \leqslant n-1$ *and let* $\lim_{n\to\infty} a_{n,k} = 0$, *then* $a_{n,k} \geqslant 0$ *for all n, k.*

Proof. We have for $k \leqslant n-1$,

$$a_{n,k} = \sum_{r=n+1}^{\infty} (a_{r-1,k} - a_{r,k})$$
$$\geqslant 0,$$

as $a_{n-1,k} \geqslant a_{n,k}$ for all n, so that $a_{r-1,k} - a_{r,k} \geqslant 0$ for all r, k. Hence $a_{n,k} \geqslant 0$ for all n, k.

Lemma 4. *If* $a_{n-1,k} \geqslant a_{n,k+1} \geqslant a_{n,k}$ *for all n, k* with $k \leqslant n-1$ *and* $\lim_{n\to\infty} a_{n,k} = 0$, *then* $\{n a_{n,0}\}$ *is bounded.*

Proof. $|n a_{n,0}| = n a_{n,0}$,

$$\leqslant \sum_{k=0}^{n} a_{n,k} \leqslant A_{n,0} \leqslant 1.$$

Thus $\{n a_{n,0}\}$ is bounded.

Lemma 5. *If*

$$g(n,t) = \frac{2}{\pi} \sum_{k=1}^{n} a_{n,k}\, k\, a_{k,0} \sin kt$$

then

$$|g(n,t)| = \begin{cases} 0(nt), & \textit{for all } t; \\ 0(A_{n,n-\tau}) & \textit{for } t \geqslant \dfrac{1}{n}. \end{cases}$$

Proof.

$$|g(n,t)| = \frac{2}{\pi}\left| \sum_{k=1}^{n} a_{n,k} k a_{k,0} \sin kt \right|$$

$$\leqslant \sum_{k=1}^{n} a_{n,k}(kt) \text{ as } ka_{k,0} \text{ is bounded}$$

$$= 0(nt), \text{ for all } t.$$

For $t \geqslant \frac{1}{n}$, we have

$$|g(n,t)| = \frac{2}{\pi}\left| \sum_{k=1}^{n} a_{n,k} k a_{k,0} \sin kt \right|$$

$$= \frac{2}{\pi}\left| \sum_{k=1}^{n} a_{n,k} \sin \mathrm{kt} \right| \text{ by using Abel's lemma.}$$

$$= 0(A_{n,n-\tau}) \text{ using Lemmas 1 and 4.}$$

5. Proof of the theorem

$$T_n = \sum_{k=1}^{n} a_{n,k}(k+1) k a_{k,0} A_k(x)$$

$$= \sum_{k=1}^{n} a_{n,k} k(k+1) a_{k,0} \frac{2}{\pi} \int_0^{\pi} \phi(t) \cos kt \, \mathrm{d}t$$

$$= \sum_{k=1}^{n} a_{n,k} k(k+1) a_{k,0} \frac{2}{\pi} \int_0^{\pi} \mathrm{d}\phi(t) \frac{\sin kt}{k}$$

$$= \sum_{k=1}^{n} a_{n,k} k a_{k,0} \frac{2}{\pi} \int_0^{\pi} \mathrm{d}\phi(t) \sin kt$$

$$= \int_0^{\pi} \mathrm{d}\phi(t) g(n,t).$$

Using lemma 2, our theorem is established if we can show that

$$\sum_{n=1}^{\infty} \frac{|T_n|}{n+1} < \infty.$$

Since $\phi(t) \epsilon B \cdot V(0,\pi)$, $\int_0^{\pi} |d\phi(t)| < \infty$. Hence our theorem will be established if we can show that

$$\sum_{n=1}^{\infty} \frac{|g(n,t)|}{n+1} < \infty.$$

Now,

$$\sum_{n=1}^{\infty} \frac{|g(n,t)|}{n+1} = \sum_{n=1}^{\tau} \frac{|g(n,t)|}{n+1} + \sum_{n=\tau+1}^{\infty} \frac{|g(n,t)|}{n+1}$$

$$= \Sigma_1' + \Sigma_2', \text{ say.}$$

$$\Sigma_1' = \sum_{n=1}^{\tau} \frac{|g(n,t)|}{n+1} = \sum_{n=1}^{\tau} \frac{0(nt)}{n+1}$$

$$= 0(t) \sum_{n=1}^{\tau} \frac{n}{n+1} = 0(1).$$

and

$$\Sigma_2' = \sum_{n=\tau+1}^{\infty} \frac{|g(n,t)|}{n+1}$$

$$= \sum_{n=\tau+1}^{\infty} \frac{0(A_{n,n-\tau})}{n+1}$$

$$= 0(1) \text{ by hypothesis.}$$

Thus the proof of the theorem is complete.

6.

Remarks. Putting

$$a_{m,n} = \begin{cases} \frac{p_{m-n}}{P_m}, \quad P_m \neq 0, & \text{for } n \leqslant m \\ 0 \quad , & \text{for } n > m \end{cases}$$

in our theorem, the theorem of Singh follows.

Putting

$$a_{m,n} = \begin{cases} \frac{1}{(m-n+1)\log(m+1)}, & \text{for } n \leqslant m, \\ 0 \quad , & \text{for } n > m, \end{cases}$$

in our theorem, theorem of Varshney follows.

Further setting

$$a_{m,n} = \frac{p_{m-n} + q_m}{P_m + (m+1)q_m}$$

where $\{q_m\}$ is a positive non-decreasing sequence, we find that $a_{m,n}$ cannot be expressed as a Nörlund coefficient, although $a_{m,n}$'s do satisfy the conditions of the theorem. Hence the scope of our theorem goes beyond the reach of Nörlund summability.

In particular, taking $p_n = 1/n$ and $q_n = n$, it can easily be verified $a_{m,n}$ satisfy the condition of the theorem but $a_{m,n}$ cannot be expressed as Nörlund coefficients.

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Absolute matrix summability of a factored Fourier series

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Abstract. A theorem for a factored Fourier series is proved.

Keywords. Absolute matrix summability; factored Fourier series; infinite triangular matrix; bounded variation.

1. Introduction

Let $\{s_n\}$ be the partial sums of an infinite series Σu_n. If $T = (a_{n,k})$ be an infinite matrix with real or complex elements, then the transform $\{t_n\}$ of $\{s_n\}$ is given by the matrix multiplication,

$$t_n = \sum_{k=0}^{\infty} a_{n,k} s_k, \tag{1}$$

(assuming that t_n exists for every $n \geqslant 0$). Now, if

$$\lim_{n\to\infty} t_n = s,$$

then the sequence $\{s_n\}$ or the series Σu_n is said to be summable $(a_{n,k})$ or T-summable to s. If $\{t_n\}$ is of bounded variation, that is to say,

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty, \tag{2}$$

the series Σu_n is said to be absolutely summable by T-method, or simply summable $|T|$.

If the matrix elements $a_{n,k} = 0$ for every $k > n$, then the matrix is called a triangular matrix.

In particular, if

$$a_{n,k} = \begin{cases} \dfrac{p_{n-k}}{P_n}, & \text{for } k \leqslant n; \\ 0, & \text{for } k > n; \end{cases}$$

where $P_n = \sum_{r=0}^{n} p_r \neq 0 (n \geqslant 0)$ then t_n defined by (1) is same as the Nörlund mean generated by the sequence of coefficients $\{p_n\}$ usually denoted as (N, p_n) mean.

Similarly, if A_n^α, $\alpha > 0$, is defined by

$$A_n^\alpha = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{1,2,3\cdots n}$$

and

$$a_{n,k} = \begin{cases} \dfrac{A_{n-k}^{\alpha-1}}{A_n^\alpha}, \quad \alpha > 0, & \text{for } k \leqslant n; \\ 0, & \text{for } k > n; \end{cases}$$

then t_n is same as the (C, α) mean, the familiar Cesàro mean of order α.

2. Notations

Let $f(t)$ be a periodic function with period 2π and integrable L over $(-\pi, \pi)$. Let

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t) \tag{3}$$

be the Fourier series of $f(t)$. We use the following notations:

$$\Phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\};$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma\alpha}\int_0^t (t-u)^{\alpha-1}\Phi(u)\mathrm{d}u, \quad \alpha > 0, \quad t > 0;$$

$$\Phi_0(t) = \phi(t);$$

$$\phi_\alpha(t) = \Gamma(1+\alpha)t^{-\alpha}\Phi_\alpha(t), \quad \alpha \geqslant 0, \quad t > 0;$$

$$g(n,t) = \frac{2}{\pi}\sum_{k=1}^{n} \frac{a_{n,k}\lambda_k}{a_{k,k}k^\alpha}\sin kt;$$

$$J(n,u) = \frac{1}{\Gamma(1-\alpha)}\int_u^\pi (t-u)^{-\alpha}\left\{\frac{\mathrm{d}}{\mathrm{d}t}g(n,t)\right\}\mathrm{d}t, \quad 0 < \alpha < 1, \quad u \geqslant 0;$$

$$V(n,u) = \frac{1}{\Gamma(1+\alpha)}\int_0^u v^\alpha \frac{\mathrm{d}}{\mathrm{d}v}J(n,v)\mathrm{d}v;$$

$$\mu_n = \sum_{k=1}^{n} a_{n,k}\lambda_k$$

$$\Delta u_n = u_n - u_{n+1},$$

$$\tau = \left[\frac{1}{t}\right];$$

and $[x]$ denotes the largest integer $\leqslant x$; and B is taken for absolute constants not necessarily the same in each occurrence.

3. Theorem

In 1957, Prasad and Bhatt [5], considered the summability $|C,\alpha|$ of the factored Fourier series under the condition that $\phi_\alpha(t)$, $0 \leqslant \alpha \leqslant 1$, is of bounded variation in $(0, \pi)$. They proved the following:

Theorem 1. *If* $\{\lambda_n\}$ *is a convex sequence such that* $\Sigma \lambda_n/n < \infty$, *and* $\phi_\alpha(t)$, $0 \leqslant \alpha \leqslant 1$, *in of bounded variation in* $(0, \pi)$, *then the series* $\Sigma A_n(t)\lambda_n$, at $t = x$, *is summable* $|C,\alpha|$.

Recently Lal [3] (for $\alpha = 1$) and Bhatt [1] (for $0 < \alpha < 1$) established the analogue of the above theorem for the absolute harmonic summability and proved results which together may be stated in the following form:

Theorem 2. *Let* $\{\lambda_n\}$ *be a convex sequence such that* $\Sigma \lambda_n/n < \infty$. *If also* $\phi_\alpha(t)$, $0 < \alpha \leqslant 1$, *is of bounded variation in* $(0, \pi)$, *then the factored Fourier series*

$$\sum \frac{A_n(t)\log(n+1)\lambda_n}{n^\alpha}$$

is harmonic summable, at $t = x$.

Our aim in the present investigations is to extend the above-mentioned results to the general framework of absolute matrix summability. In what follows, we establish the following result:

Theorem. *Let* $|T| = (a_{n,k})$ *be an infinite triangular matrix with* $\lim_{n \to \infty} a_{n,k} = 0$. *Further let us write* $\sum_{r=k}^{n} a_{n,r} = A_{n,k}$ *and assume that* $A_{n,0} = 1$ *for every* $n \geqslant 0$. *If* $\phi_\alpha(t)$, $0 < \alpha < 1$, *is a function of bounded variation in* $(0, \pi)$, *then the factored Fouries series* $\sum \frac{A_n(t)\lambda_n}{n^\alpha a_{n,n}}$ *is summable* $|T|$, *provided that* $\{\lambda_n\}$ *is a convex sequence for which* $\sum \frac{\lambda_n}{n} < \infty$, *and*

$$a_{n-1,k} \geqslant a_{n,k+1} \geqslant a_{n,k} \quad \textit{for all } n, k \text{ with } k \leqslant n-1 \tag{4}$$

$$(n+1)(a_{n-1,k} - a_{n,k+1}) \geqslant \Delta_k\{(n-k)a_{n,k}\} \tag{5}$$

$$\sum \frac{a_{n,0}}{n+1} < \infty \tag{6}$$

$$\left\{\frac{1}{n^\alpha a_{n,n}}\right\}, \quad 0 < \alpha \leqslant 1, \textit{ is monotonically non-increasing} \tag{7}$$

and

$$\left\{\frac{Q_n}{n^\alpha}\right\} \quad \textit{is monotonic, where} \tag{8}$$

$$Q_n = \sum_{r=1}^{n} \frac{\lambda_r}{r^\alpha a_{r,r}},$$

4. Lemmas

We require the following lemmas for the proof of our theorem.

Lemma 1. [4] *If* $\{a_{n,k}\}_{k=0}^{n}$ *is a non-negative and non-decreasing sequence with respect to k, then for* $0 \leqslant a < b \leqslant \infty$, $0 \leqslant t \leqslant \pi$ *and for every n*

$$\left| \sum_{k=a}^{b} a_{n,n-k} \exp[i(n-k)t] \right| \leqslant BA_{n,n-\tau},$$

where B is an absolute constant and $\tau = [1/t]$.

Lemma 2. [4] *If* $\{a_{n,k}\}_{k=0}^{n}$ *is a non-negative and non-decreasing sequence with respect to k such that* $A_{n,0} = 1$ *then as* $n \to \infty$, $a_{n,k} = O\left(\frac{1}{n-k+1}\right)$ *for every* $k \leqslant n$.

Lemma 3. [2] *Let conditions* (4)–(6) *of the theorem hold. If the sequence of partial sums of* Σu_n *be bounded, then the necessary and sufficient condition for it to be summalbe* $|T|$, *is that*

$$\sum_{n=1}^{\infty} \frac{1}{n+1} \left| \sum_{k=0}^{n} (k+1) a_{n,k} u_k \right| < \infty.$$

Lemma 4. *If* $\{\lambda_n\}$ *is a convex sequence such that* $\Sigma \lambda_n / n < \infty$, *and*

$$\mu_n = \sum_{k=1}^{n} a_{n,k} \lambda_k,$$

then

$$\sum_{n=1}^{\infty} \frac{\mu_n}{n+1} < \infty$$

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\mu_n}{n+1} &= \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{k=1}^{n} a_{n,k} \lambda_k \\ &= \sum_{n=1}^{\infty} \frac{1}{n+1} \left[\sum_{k=1}^{m} a_{n,k} \lambda_k + \sum_{k=m+1}^{n} a_{n,k} \lambda_k \right] \\ &= \Sigma_1 + \Sigma_2, \text{ say.}\end{aligned}$$

Now,

$$\begin{aligned}\Sigma_1 &\leqslant \sum_{n=1}^{\infty} \frac{1}{n+1} a_{n,m} \sum_{k=1}^{m} \lambda_k \\ &\leqslant A \sum_{m=1}^{\infty} \frac{1}{m+1} a_{n,m} \sum_{k=1}^{m} \lambda_k \\ &= A \sum_{k=1}^{\infty} \lambda_k \sum_{m=k}^{\infty} \frac{1}{m(m+1)} \\ &= A \sum_{k=1}^{\infty} \frac{\lambda_k}{k}\end{aligned}$$

and

$$\begin{aligned}\Sigma_2 &\leqslant \sum_{n=1}^{\infty}\frac{1}{n+1}\sum_{k=m+1}^{n} a_{n,k}\lambda_k\\ &\leqslant \sum_{n=1}^{\infty}\frac{1}{n+1}\lambda_m\sum_{k=m+1}^{n} a_{n,k}\\ &\leqslant B\sum_{n=1}^{\infty}\frac{\lambda_m}{m}\\ &<\infty,\end{aligned}$$

and the lemma is established.

Lemma 5. *If*

$$g(n,t)=\frac{2}{\pi}\sum_{k=1}^{n}\frac{a_{n,k}}{a_{k,k}}\frac{\lambda_k}{k^\alpha}\sin kt$$

then

$$|g(n,t)|=\begin{cases}O(\mu_n), & \textit{for all } t;\\ O(Q_\tau a_{n,m})+O\left(A_{n,n-\tau}\dfrac{\lambda_n}{a_{n,n}n^\alpha}\right), & \textit{for } t\geqslant \dfrac{1}{n}.\end{cases}$$

and

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}g(n,t)\right|=\begin{cases}O(n\mu_n), & \textit{for all } t;\\ O(nQ_\tau a_{n,m})+O\left(A_{n,n-\tau}\dfrac{\lambda_n}{\alpha_{n,n}n^{\alpha-1}}\right), & \textit{for } t\geqslant \dfrac{1}{n}.\end{cases}$$

Proof. For all t

$$\begin{aligned}|g(n,t)| &=\frac{2}{\pi}\left|\sum_{k=1}^{n}\frac{a_{n,k}}{a_{k,k}}\frac{\lambda_k}{k^\alpha}\sin kt\right|\\ &\leqslant B\left|\frac{2}{\pi}\sum_{k=1}^{n}a_{n,k}\lambda_k\right|\\ &=O(\mu_n).\end{aligned}$$

For $t\geqslant \frac{1}{n}$,

$$\begin{aligned}|g(n,t)| &\leqslant \frac{2}{\pi}\left[\left|\sum_{k=1}^{m}\right|+\left|\sum_{k=m+1}^{n}\right|\right]\\ &=\frac{2}{\pi}[S_1+S_2], \quad \text{say.}\end{aligned}$$

Now,

$$S_1=\left|\sum_{k=1}^{m-1}\Delta_k a_{n,k}\sum_{r=1}^{k}\frac{\lambda_r\sin rt}{r^\alpha a_{r,r}}+a_{n,m}\sum_{r=1}^{k}\frac{\lambda_r\sin rt}{r^\alpha a_{r,r}}\right|$$

$$\begin{aligned}&\leqslant BQ_\tau\left[\sum_{k=1}^{m-1}|\Delta_k a_{n,k}|+a_{n,m}\right]\\ &=O(Q_\tau a_{n,m})\end{aligned}$$

and,

$$S_2 \leqslant \sum_{k=m+1}^{n-1} \Delta\left(\frac{\lambda_k}{a_{k,k}k^\alpha}\right) \sum_{r=0}^{k} a_{n,r} \sin rt + \frac{\lambda_n}{a_{n,n}n^\alpha} \sum_{r=0}^{n} a_{n,r} \sin rt$$

$$- \frac{\lambda_{m+1}}{a_{m+1,m+1}(m+1)^\alpha} \sum_{r=0}^{m} a_{n,r} \sin rt$$

$$= O\left(A_{n,n-\gamma}\frac{\lambda_n}{a_{n,n}n^\alpha}\right), \quad \text{using lemma 1.}$$

Now proceeding as above, we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} g(n,t)\right| = \frac{2}{\pi}\left|\sum_{k=1}^{n} \frac{a_{n,k}}{a_{k,k}} \frac{\lambda_k}{k^{\alpha-1}} \cos kt\right|$$

Hence for all t

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} g(n,t)\right| \leqslant O(n) \sum_{k=1}^{n} a_{n,k}\lambda_k$$

$$= O(n\mu_n)$$

and for $t \geqslant \frac{1}{n}$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} g(n,t)\right| = O(nQ_\tau a_{n,m}) + O\left(\frac{A_{n,n-\tau}\lambda_n}{a_{n,m}n^{\alpha-1}}\right)$$

Lemma 6. *If*

$$J(n,u) = \frac{1}{\Gamma(1-\alpha)} \int_u^\pi (t-u)^{-\alpha} \left\{\frac{\mathrm{d}}{\mathrm{d}t} g(n,t)\right\} \mathrm{d}t$$

then

$$|J(n,u)| = \begin{cases} O(n^\alpha \mu_n), & \textit{for all } u; \\ O(n^\alpha Q_{[1/u]} a_{n,m}) + O\left(\dfrac{A_{n,n-\nu}\lambda_n}{a_{n,n}}\right), & \textit{for } u \geqslant \dfrac{1}{n} \textit{ and } U = [1/u] \end{cases}$$

Proof.

$$|J(n,u)| = \frac{1}{\Gamma(1-\alpha)} \int_u^\pi (t-u)^{-\alpha} \left\{\frac{\mathrm{d}}{\mathrm{d}t} g(n,t)\right\} \mathrm{d}t$$

$$= \int_u^{u+(1/n)} + \int_{u+(1/n)}^{\pi}$$

$$= J_1 + J_2, \text{ say.}$$

For all u

$$|J_1| \leqslant \frac{1}{\Gamma(1-\alpha)} \int_u^{u+(1/n)} (t-u)^{-\alpha} \left|\frac{\mathrm{d}}{\mathrm{d}t} g(n,t)\right| \mathrm{d}t$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_u^{u+(1/n)} (t-u)^{-\alpha} O(n\mu_n) \mathrm{d}t, \quad \text{using lemma 5.}$$

$$= O(n\mu_n) \int_u^{u+(1/n)} (t-u)^{-\alpha} \mathrm{d}t$$

$$= O(n^\alpha \mu_n) \quad \text{and}$$

$$|J_2| = \frac{1}{\Gamma(1-\alpha)} \left| \int_{u+(1/n)}^{\pi} (t-u)^{-\alpha} \left\{ \frac{\mathrm{d}}{\mathrm{d}t} g(n,t) \right\} \mathrm{d}t \right|$$

$$= O(n^\alpha) \left| \int_{u+(1/n)}^{\eta} \frac{\mathrm{d}}{\mathrm{d}t} g(n,t) \mathrm{d}t \right|, \quad \text{where } u + \frac{1}{n} < \eta < \pi$$

$$= O(n^\alpha) \{ |g(n,t)| \}_{u+(1/n)}^{\eta}$$

$$= O(n^\alpha \mu_n), \quad \text{using lemma 5.}$$

For $u \geqslant \frac{1}{n}$

$$|J_1| \leqslant \frac{1}{\Gamma(1-\alpha)} \int_u^{u+(1/n)} (t-u)^{-\alpha} \left| \frac{\mathrm{d}}{\mathrm{d}t} g(n,t) \right| \mathrm{d}t$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_u^{u+(1/n)} (t-u)^{-\alpha} \left\{ O(nQ_\tau a_{n,m}) + O\left(\frac{A_{n,n-\tau} \lambda_n}{a_{n,n} n^{\alpha-1}} \right) \right\} \mathrm{d}t,$$

using lemma 5

$$= \left\{ O(nQ_{[1/u]} a_{n,m}) + O\left(\frac{A_{n,n-[1/u]} \lambda_n}{a_{n,n} n^{\alpha-1}} \right) \right\} \int_u^{u+(1/n)} (t-u)^{-\alpha} \mathrm{d}t$$

$$= O(n^\alpha Q_{[1/u]} a_{n,m}) + O\left(\frac{A_{n,n-[1/u]} \lambda_n}{a_{n,n}} \right),$$

and

$$|J_2| = \left| \frac{1}{\Gamma(1-\alpha)} \int_{u+(1/n)}^{\pi} (t-u)^{-\alpha} \left\{ \frac{\mathrm{d}}{\mathrm{d}t} g(n,t) \right\} \mathrm{d}t \right|$$

$$= \left| \frac{n^\alpha}{\Gamma(1-\alpha)} \int_{u+(1/n)}^{\eta} \left\{ \frac{\mathrm{d}}{\mathrm{d}t} g(n,t) \right\} \mathrm{d}t \right|, \quad \text{where } u + \frac{1}{n} < \eta < \pi$$

$$= O(n^\alpha) \{ |g(n,t)| \}_{u+(1/n)}^{\eta}$$

$$= O(n^\alpha Q_{[1/u]} a_{n,m}) + O\left(\frac{A_{n,n-[1/u]} \lambda_n}{a_{n,m}} \right), \quad \text{by lemma 5.}$$

Lemma 7. *If*

$$V(n,u) = \frac{1}{\Gamma(1+\alpha)} \int_0^u v^\alpha \frac{\mathrm{d}}{\mathrm{d}v} J(n,v) \mathrm{d}v$$

then

$$V(n,\pi) = 0$$

Proof.

$$V(n,\pi) = \frac{1}{\Gamma(1+\alpha)} \int_0^\pi v^\alpha \frac{\mathrm{d}}{\mathrm{d}v} J(n,v) \mathrm{d}v$$

$$
\begin{aligned}
&= \frac{1}{-\Gamma\alpha}\int_0^{\pi} v^{\alpha-1} J(n,v)\,\mathrm{d}v \\
&= -\frac{1}{\Gamma(1-\alpha)\Gamma\alpha}\int_0^{\pi} v^{\alpha-1}\,\mathrm{d}v \int_u^{\pi} (t-v)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t} g(n,t)\,\mathrm{d}t \\
&= -\frac{1}{\Gamma\alpha\Gamma(1-\alpha)}\int_0^{\pi} g(n,t)\,\mathrm{d}t \int_0^{t} (t-v)^{-\alpha} v^{\alpha-1}\,\mathrm{d}v \\
&= -\frac{1}{\Gamma\alpha\Gamma(1-\alpha)}\int_0^{\pi} \frac{\mathrm{d}}{\mathrm{d}t} g(n,t)\,\mathrm{d}t \int_0^{1} (1-x)^{-\alpha} x^{\alpha-1}\,\mathrm{d}x \\
&= -\int_0^{\pi} \frac{\mathrm{d}}{\mathrm{d}t} g(n,t)\,\mathrm{d}t \\
&= g(n,0) - g(n,\pi) \\
&= 0.
\end{aligned}
$$

Lemma 8. *If*

$$
V(n,u) = \frac{1}{\Gamma(1+\alpha)}\int_0^{u} v^{\alpha}\frac{\mathrm{d}}{\mathrm{d}v} J(n,v)\,\mathrm{d}v
$$

then

$$
|V(n,u)| = \begin{cases} O(u^{\alpha} n^{\alpha} \mu_n), & \textit{for all } u, \\ O(n^{\alpha} u^{\alpha} Q_{[1/u]} a_{n,m}) + O\left(\dfrac{A_{n,n-v}\lambda_n}{a_{n,m}} u^{\alpha}\right), & \textit{for } u \geqslant \dfrac{1}{n}. \end{cases}
$$

Proof. We have

$$
\begin{aligned}
V(n,u) &= \frac{1}{\Gamma(1+\alpha)}\int_0^{u} v^{\alpha}\frac{\mathrm{d}}{\mathrm{d}v} J(n,v)\,\mathrm{d}v \\
&= \left[\frac{u^{\alpha} J(n,u)}{\Gamma(1+\alpha)}\right] - \frac{1}{\Gamma\alpha}\int_0^{u} v^{\alpha-1} J(n,v)\,\mathrm{d}v \\
&= O(n^{\alpha} u^{\alpha} \mu_n), \quad \text{using lemma 6.}
\end{aligned}
$$

Again, for $u \geqslant \dfrac{1}{n}$

$$
\begin{aligned}
V(n,\pi) - V(n,u) &= \frac{1}{\Gamma(1+\alpha)}\int_u^{\pi} v^{\alpha}\frac{\mathrm{d}}{\mathrm{d}v} J(n,v)\,\mathrm{d}v \\
&= \left[\frac{v^{\alpha} J(n,v)}{\Gamma(1+\alpha)}\right]_u^{\pi} - \frac{1}{\Gamma\alpha}\int_u^{\pi} v^{\alpha-1} J(n,v)\,\mathrm{d}v.
\end{aligned}
$$

Hence

$$
\begin{aligned}
|V(n,u)| &= \left| O - \left[\frac{v^{\alpha} J(n,v)}{(1+\alpha)}\right]_u^{\pi} + \frac{1}{\Gamma\alpha}\int_u^{\pi} v^{\alpha-1} J(n,v)\,\mathrm{d}v \right|, \quad \text{using lemma 7} \\
&\leqslant O(u^{\alpha} n^{\alpha} Q_{[1/u]} a_{n,m}) + O\left(\frac{A_{n,n-v}\lambda_n}{a_{n,m}} u^{\alpha}\right) + \left|\frac{1}{\Gamma\alpha}\int_u^{\pi} v^{\alpha-1} J(n,v)\,\mathrm{d}v\right| \\
&\qquad \text{using lemma 6.}
\end{aligned}
$$

Now,

$$\left|\frac{1}{\Gamma\alpha}\int_u^\pi v^{\alpha-1}J(n,v)\mathrm{d}v\right| = \left|\frac{1}{\Gamma\alpha}\int_u^\pi v^{\alpha-1}\left[\frac{1}{\Gamma(1-\alpha)}\int_u^\pi (t-v)^{-\alpha}\right.\right.$$

$$\left.\left.\times\left\{\frac{\mathrm{d}}{\mathrm{d}t}g(n,t)\right\}\mathrm{d}t\right]\mathrm{d}v\right|$$

$$= \left|\frac{1}{\Gamma\alpha\Gamma(1-\alpha)}\int_u^\pi\left(\frac{\mathrm{d}}{\mathrm{d}t}g(n,t)\right)\mathrm{d}t\int_u^t (t-u)^{-\alpha}u^{\alpha-1}\mathrm{d}v\right|$$

$$= \frac{1}{\Gamma\alpha\Gamma(1-\alpha)}\left|\int_u^\pi\left(\frac{\mathrm{d}}{\mathrm{d}t}g(n,t)\right)\mathrm{d}t\int_{u/t}^1 (1-v)^{-\alpha}v^{\alpha-1}\mathrm{d}v\right|$$

$$\leqslant \frac{\beta(\alpha,1-\alpha)}{\Gamma\alpha\Gamma(1-\alpha)}\left|\int_u^\pi\left(\frac{\mathrm{d}}{\mathrm{d}t}g(n,t)\right)\mathrm{d}t\right|$$

$$= O[|g(n,u)|]$$

$$= O(Q_{[1/u]}a_{n,m}) + O\left(A_{n,n-[1/u]}\frac{\lambda_n}{a_{n,n}n^\alpha}\right),$$

using lemma 5.

$$\leqslant O(n^\alpha u^\alpha Q_{[1/u]}a_{n,m}) + O\left(\frac{A_{n,n-U}\lambda_n u^\alpha}{a_{n,n}}\right)$$

This completes the proof of the lemma.

5. *Proof of the theorem.* Let

$$\tau_n = \sum_{k=1}^n a_{n,k}(k+1)\frac{A_k(t)\lambda_k}{k^\alpha a_{k,k}}$$

It is known [6] that if $\Phi_\alpha(t)$, $0 \leqslant \alpha < 1$, is of bounded variation, the Fourier series of $f(t)$ is convergent. Using hypothesis (4) to (6) and lemma 3 to prove the theorem we show that

$$\sum_{n=1}^\infty \frac{|\tau_n|}{n+1} < \infty.$$

Now

$$A_k(t) = \frac{2}{\pi}\int_0^\pi \phi(t)\cos kt\,\mathrm{d}t.$$

We have

$$\tau_n = \frac{2}{\pi}\sum_{k=1}^n a_{n,k}(k+1)\frac{\lambda_k}{k^\alpha a_{k,k}}\int_0^\pi \phi(t)\cos kt\,\mathrm{d}t$$

$$= \int_0^\pi \phi(t)\frac{\mathrm{d}}{\mathrm{d}t}g(n,t)\mathrm{d}t$$

$$= \int_0^\pi \frac{\mathrm{d}}{\mathrm{d}t}g(n,t)\left\{\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-u)^{-\alpha}\mathrm{d}\Phi_\alpha(u)\right\}\mathrm{d}t$$

$$= \int_0^\pi \mathrm{d}\Phi_\alpha(u)\frac{1}{\Gamma(1-\alpha)}\int_u^\pi (t-u)^{-\alpha}\left\{\frac{\mathrm{d}}{\mathrm{d}t}g(n,t)\right\}\mathrm{d}t$$

$$= \int_0^\pi J(n,u)\,\mathrm{d}\Phi_\alpha(u)$$

$$= \Phi_\alpha(\pi)J(n,\pi) - \int_0^\pi \Phi_\alpha(u)\frac{\mathrm{d}}{\mathrm{d}u}J(n,u)\,\mathrm{d}u$$

$$= \frac{1}{\Gamma(1+\alpha)}\int_0^\pi u^\alpha \phi_\alpha(u)\frac{\mathrm{d}}{\mathrm{d}u}J(n,u)\,\mathrm{d}u$$

$$= \left[\frac{1}{\Gamma(1+\alpha)}\phi_\alpha(u)\int_0^u v^\alpha \frac{\mathrm{d}}{\mathrm{d}v}J(n,v)\,\mathrm{d}v\right]_0^\pi$$

$$= \frac{1}{\Gamma(1+\alpha)}\int_0^\pi \mathrm{d}\phi_\alpha(u)\int_0^u v^\alpha \frac{\mathrm{d}}{\mathrm{d}v}J(n,v)\,\mathrm{d}v$$

$$= \phi_\alpha(\pi)\,V(n,\pi) - \int_0^\pi V(n,u)\,\mathrm{d}\phi_\alpha(u).$$

Collecting our results, we get

$$\tau_n = \phi_\alpha(\pi)J(n,\pi) - \phi_\alpha(\pi)\,V(n,\pi) + \int_0^\pi V(n,u)\,\mathrm{d}\phi_\alpha(u)$$

$$= O - \phi_\alpha(\pi)O + \int_0^\pi V(n,u)\,\mathrm{d}\phi_\alpha(u)$$

$$= \int_0^\pi V(n,u)\,\mathrm{d}\phi_\alpha(u).$$

Now, for $0<\alpha<1$,

$$\sum_{n=1}^\infty \frac{|\tau_n|}{n+1} \leqslant \sum_{n=1}^\infty \frac{1}{n+1}\left|\int_0^\pi V(n,u)\Phi_\alpha(u)\right|$$

$$\leqslant \sum_{n=1}^\infty \frac{1}{n+1}\left\{\left|\int_0^{1/n}\right| + \left|\int_{1/n}^\pi\right|\right\}$$

$$= I_1 + I_2, \quad \text{say.}$$

where

$$I_1 = \sum_{n=1}^\infty \frac{1}{n+1}\int_0^{1/n} O(n\alpha u^\alpha \mu_n)|\mathrm{d}\phi_\alpha(u)|, \quad \text{using lemma 8}$$

$$\leqslant O\sum_{n=1}^\infty \frac{\mu_n}{n+1}\int_0^{1/n}|\mathrm{d}\phi_\alpha(u)|$$

$= O(1)$ using lemma 4 and hypothesis.

$$I_2 = \sum_{n=1}^\infty \frac{1}{n+1}\left|\int_{1/n}^\pi V(n,u)\,\mathrm{d}\phi_\alpha(u)\right|$$

$$\leqslant \sum_{n=1}^\infty \frac{1}{n+1}\left|\int_{1/n}^\pi \left\{O(u^\alpha n^\alpha Q_{[1/u]}a_{n,m}) + O\left(\frac{A_{n,n-U}\lambda_n u^\alpha}{a_{n,n}}\right)\right\}|\mathrm{d}\phi_\alpha(u)|\right.$$

using lemma 8.

$$= I_{2,1} + I_{2,2} \quad \text{say.}$$

Since $u^\alpha Q_{[1/u]}$ is monotonic, we have

$$
\begin{aligned}
I_{2,1} &= \sum_{n=1}^{\infty} \frac{1}{n+1} \int_{1/n}^{\pi} O(n^\alpha u^\alpha Q_{[1/u]} a_{n,m}) |\mathrm{d}\phi_\alpha(u)| \\
&= \sum_{n=1}^{\infty} O \frac{(n^\alpha a_{n,m})}{n+1} \left\{ \frac{Q_n}{n\alpha} \int_{1/n}^{\xi} |\mathrm{d}\phi_\alpha(u)| + \pi^\alpha Q_{[1/n]} \int_{\xi}^{\pi} |\mathrm{d}\phi_\alpha(u)| \right\} \left(\frac{1}{n} \leqslant \xi \leqslant \pi \right) \\
&= O\left(\sum_{n=1}^{\infty} \frac{Q_n a_{n,m}}{n+1} \right) + O\left(\sum_{n=1}^{\infty} \frac{n^\alpha a_{n,m}}{n+1} \right) \quad \text{using lemma 2.} \\
& O\left(\sum_{n=1}^{\infty} \frac{Q_n}{n(n+1)} \right) + O\left(\sum_{n=1}^{\infty} \frac{1}{(n+1)} \frac{1}{n^{1-\alpha}} \right) \\
&= O\left(\sum_{r=1}^{\infty} \frac{\lambda_r}{r^\alpha a_{r,r}} \sum_{n=r}^{\infty} \frac{1}{n(n+1)} \right) + O(1) \\
&= O\left(\sum_{r=1}^{\infty} \lambda_r / r \right) + O(1) \\
&= O(1).
\end{aligned}
$$

Since $\left\{ \frac{1}{n^\alpha a_{n,n}} \right\}$ is monotonically non-increasing, we have,

$$
\begin{aligned}
I_{2,2} &= \sum_{n=1}^{\infty} \frac{1}{n+1} O(\lambda_n) \int_{1/n}^{\xi'} |\mathrm{d}\phi_\alpha(u)| \left(\frac{1}{n} \leqslant \xi' \leqslant \pi \right) \\
&= O\left(\sum_{n=1}^{\infty} \frac{\lambda_n}{n} \right) \\
&= O(1).
\end{aligned}
$$

This completes the proof of the theorem.

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Multipliers for $|\bar{N},p_n;\delta|_k$ summability of infinite series

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Abstract. In this paper a theorem on $|\bar{N},p_n;\delta|_k$ summability factors, which generalizes a theorem of Bor [3] on $|\bar{N},p_n|_k$ summability factors, has been proved.

Keywords. Summability factors; infinite series.

1. Introduction

Let Σa_n be a given infinite series with partial sums (s_n) and let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geqslant 1). \tag{1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (t_n) of the $(\bar{N},p_n)$ means of the sequence (s_n), generated by the sequence of coefficients (p_n). The series Σa_n is said to be summable $|\bar{N},p_n|_k$, $k \geqslant 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |t_n - t_{n-1}|^k < \infty \tag{3}$$

and it is said to be summable $|\bar{N},p_n;\delta|_k$, $k \geqslant 1$ and $\delta \geqslant 0$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k + k - 1} |t_n - t_{n-1}|^k < \infty. \tag{4}$$

In the special case when $p_n = 1$ for all values of n (resp. $\delta = 0$) $|\bar{N},p_n;\delta|_k$ summability is the same as $|C,1;\delta|_k$ (resp. $|\bar{N},p_n|_k$) summability.

2.

The author proved the following theorem for $|\bar{N},p_n|_k$ summability (see [3]).

Theorem A. *Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \text{ as } n \to \infty. \tag{5}$$

If (X_n) is a positive monotonic nondecreasing sequence such that

$$\lambda_n X_n = O(1) \text{ as } n \to \infty, \tag{6}$$

$$\sum_{v=1}^{n} vX_v |\Delta^2 \lambda_v| = O(1), \tag{7}$$

$$\sum_{v=1}^{n} \frac{p_v}{P_v} |t_v|^k = O(X_n) \text{ as } n \to \infty, \tag{8}$$

then the series $\Sigma a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geqslant 1$.

If we take $p_n = 1$ for all values of n in this theorem, then we get a result due to Mazhar [5] for $|C, 1|_k$ summability methods.

3.

The aim of this paper is to generalize Theorem A for $|\bar{N}, p_n; \delta|_k$ summability in the form of the following theorem.

Theorem. *Let (p_n) be a sequence of positive numbers such that condition* (5) *of Theorem A is satisfied. If (X_n) is a positive monotonic nondecreasing sequence such that conditions* (6), (7) *of Theorem A are satisfied and*

$$\sum_{v=1}^{n} (P_v/p_v)^{\delta k - 1} |t_v|^k = O(X_n) \text{ as } n \to \infty, \tag{9}$$

$$\sum_{n=v}^{\infty} (P_n/p_n)^{\delta k - 1} \frac{1}{P_{n-1}} = O\left\{ (P_v/p_v)^{\delta k} \frac{1}{P_v} \right\}, \tag{10}$$

then the series $\Sigma a_n \lambda_n$ is summable $|\bar{N}, p_n; \delta|_k$ for $k \geqslant 1$ and $\delta \geqslant 0$.

It should be noted that if we take $\delta = 0$ in this theorem, then we get Theorem A. Because in this case condition (9) reduces to condition (8) and condition (10) is obvious.

4.

We need the following lemma for the proof of our theorem.

Lemma ([4]). *If the condition* (7) *of Theorem A is satisfied, then*

$$\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty, \tag{11}$$

$$nX_n |\Delta \lambda_n| = O(1) \text{ as } n \to \infty. \tag{12}$$

5.

Proof of Theorem. Let (T_n) be the sequence of $(\bar{N},p_n)$ means of the series $\Sigma a_n\lambda_n$. Then, by definition, we have

$$T_n=\frac{1}{P_n}\sum_{v=0}^{n}p_v\sum_{i=0}^{v}a_i\lambda_i=\frac{1}{P_n}\sum_{v=0}^{n}(P_n-P_{v-1})a_v\lambda_v.$$

Then, for $n\geqslant 1$, we get

$$T_n-T_{n-1}=\frac{p_n}{P_nP_{n-1}}\sum_{v=0}^{n}P_{v-1}a_v\lambda_v=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}\frac{P_{v-1}\lambda_v}{v}va_v. \tag{13}$$

Applying Abel's transformation to the right hand side of (13), we have

$$\begin{aligned}T_n-T_{n-1}&=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\Delta\left(\frac{P_{v-1}\lambda_v}{v}\right)\sum_{i=1}^{v}ia_i+\frac{p_n\lambda_n}{nP_n}\sum_{v=1}^{n}va_v=\frac{(n+1)p_nt_n\lambda_n}{nP_n}\\&\quad-\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_vt_v\lambda_v\frac{v+1}{v}+\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v\Delta\lambda_vt_v\frac{v+1}{v}\\&\quad+\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v\lambda_{v+1}t_v\frac{1}{v}=T_{n,1}+T_{n,2}+T_{n,3}+T_{n,4},\text{ say.}\end{aligned}$$

To complete the proof of the theorem it is sufficient, by Minkowski's inequality, to show that

$$\sum_{n=1}^{\infty}(P_n/p_n)^{\delta k+k-1}|T_{n,i}|^k<\infty,\quad\text{for } i=1,2,3,4.$$

Firstly, we have that

$$\begin{aligned}\sum_{n=1}^{m}(P_n/p_n)^{\delta k+k-1}|T_{n,1}|^k&=O(1)\sum_{n=1}^{m}|\lambda_n|^{k-1}|\lambda_n|(P_n/p_n)^{\delta k-1}|t_n|^k\\=O(1)\sum_{n=1}^{m}|\lambda_n|(P_n/p_n)^{\delta k-1}|t_n|^k&=O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}(P_v/p_v)^{\delta k-1}|t_v|^k\\+O(1)|\lambda_m|\sum_{n=1}^{m}(P_n/p_n)^{\delta k-1}|t_n|^k&=O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n\\+O(1)|\lambda_m|X_m&=O(1)\text{ as } m\to\infty,\text{ by virtue of (6), (9) and (11).}\end{aligned}$$

Now, when $k>1$, applying Hölder's inequality, with indices k and k', where $1/k+1/k'=1$, as in $T_{n,1}$, we have that

$$\begin{aligned}\sum_{n=2}^{m+1}(P_n/p_n)^{\delta k+k-1}|T_{n,2}|^k&=O(1)\sum_{n=2}^{m+1}(P_n/p_n)^{\delta k-1}\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}p_v|t_v|^k|\lambda_v|^k\right\}\\&\quad\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}\end{aligned}$$

$$= O(1) \sum_{v=1}^{m} |\lambda_v|^{k-1} |\lambda_v| |t_v|^k \sum_{n=v+1}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-}}$$

$$= O(1) \sum_{v=1}^{m} |\lambda_v| (P_v/p_v)^{\delta k-1} |t_v|^k = O(1) \text{ as } m \to \infty.$$

Again, using the fact that $P_v = O(vp_v)$, by (5) and $v|\Delta\lambda_v| = O(1/X_v) = O(1)$, by (
we have that·

$$\sum_{n=2}^{m+1} (P_n/p_n)^{\delta k+k-1} |T_{n,3}|^k = O(1) \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} P_v |\Delta\lambda_v| |t_v| \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} v p_v |\Delta\lambda_v| |t_v| \right\}$$

$$= O(1) \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} (v|\Delta\lambda_v|)^k p_v |t_v| \right.$$

$$\times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} (v|\Delta\lambda_v|)^{k-1} v |\Delta\lambda_v| p_v |t_v|^k$$

$$\times \sum_{n=v+1}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} v |\Delta\lambda_v| (P_v/p_v)^{\delta k-1} |t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} \Delta(v|\Delta\lambda_v|) \sum_{i=1}^{v} (P_i/p_i)^{\delta k-1} |t_i|^k$$

$$+ O(1) m |\Delta\lambda_m| \sum_{v=1}^{m} (P_v/p_v)^{\delta k-1} |t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} |\Delta(v|\Delta\lambda_v|)| X_v + O(1) m |\Delta\lambda_m| X_m$$

$$= O(1) \sum_{v=1}^{m-1} v X_v |\Delta^2 \lambda_v| + O(1) \sum_{v=1}^{m-1} |\Delta\lambda_{v+1}| X_v$$

$$+ O(1) m |\Delta\lambda_m| X_m = O(1) \text{ as } m \to \infty,$$

by virtue of (7), (9), (10), (11), and (12). Finally, using the fact that $P_v = O(vp_v)$, by
as in $T_{n,1}$, we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{\delta k+k-1} |T_{n,4}|^k = O(1) \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} |\lambda_{v+1}| p_v |t_v| \right\}$$

$$= O(1) \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k - 1} \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} |\lambda_{v+1}|^k p_v |t_v|^k \right\}$$

$$\times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m-1} |\lambda_{v+1}|^{k-1} |\lambda_{v+1}| p_v |t_v|^k$$

$$\times \sum_{n=v+1}^{m+1} (P_n/p_n)^{\delta k - 1} \frac{1}{P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| (P_v/p_v)^{\delta k - 1} |t_v|^k = O(1) \text{ as } m \to \infty.$$

Therefore, we get that

$$\sum_{n=1}^{m} (P_n/p_n)^{\delta k + k - 1} |T_{n,i}|^k = O(1) \text{ as } m \to \infty, \text{ for } i = 1, 2, 3, 4.$$

This completes the proof of the theorem. If we take $p_n = 1$ for all values of n in this theorem, then we get a result concerning the $|C, 1; \delta|_k$ summability methods.

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Bishop decompositions for vector function spaces

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Abstract. Various Bishop type decompositions for vector function spaces are introduced and discussed. Conditions are given under which some of them coincide. Several examples and counter examples are also given.

Keywords. Vector function spaces; antisymmetric set; Bishop decomposition.

1. Introduction and preliminaries

The antisymmetric decomposition introduced by Bishop [1] plays a vital role in the study of function algebras. This notion was generalized for function spaces and vector function spaces by Edwards [2] and by Feyel and Pradelle [3] respectively. We extend Edwards' definition to vector function spaces and also introduce a new decomposition. We make a comparative study of these various decompositions and study them for two special classes, one of which is the standard tensor product, the other is helpful to describe vector-valued phenomena.

Let $C(X;B)$ denote the set of all continuous functions on a compact Hausdorff space X with values in a Banach space B. Then $C(X;B)$ is a Banach space with pointwise operations and the supnorm

$$\|f\| = \sup_{x\in X} \|f(x)\| \quad (f\in C(X;B)).$$

For $B=\mathbb{C}$, $C(X;B)$ is denoted by $C(X)$. For $b\in B$, the function identically equal to b is denoted by b and is called a vector constant function. If B is a Banach algebra, so is $C(X;B)$ and if B is commutative and has an identity, $C(X;B)$ also has these properties.

Now onwards X is a compact Hausdorff space, B is a commutative Banach algebra with identity e and $m(B)$ denotes its maximal ideal space.

DEFINITION 1.1.

A vector function space on X is a closed subspace of $C(X;B)$ containing vector constant functions. When $B=\mathbb{C}$, such a space is called a complex function space.

A vector function algebra on X is a closed subalgebra of $C(X;B)$ which separates the points of X and contains vector constants.

Examples 1.2 (i) Let A be a complex function space on X. For $f\in A$ and $b\in B$, $f\otimes b$ denotes the function $(f\otimes b)(x)=f(x)b$ $(x\in X)$. The tensor product $A\hat{\otimes} B$, is the closure

in $C(X;B)$ of the span of the functions $\{f\otimes b\colon f\in A, b\in B\}$. $A\hat{\otimes}B$ is a vector function space on X. If A is a function algebra on X, then $A\hat{\otimes}B$ is a vector function algebra on X.
(ii) Again let A be as above and let ϕ_0 be a fixed complex homomorphism on B i.e., $\phi_0\in m(B)$. Define

$$V=\{f\in C(X;B)\colon \phi_0 o f\in A\}.$$

Then V is a vector function space and is a vector function algebra, if A is a function algebra.

Before defining the Bishop or Šilov decompositions for vector function spaces, let us note that a decomposition of a topological space Y is a partition of Y into disjoint closed sets whose union is Y. If $\mathscr{E}$ and $\mathscr{F}$ are two decompositions of Y and if for each $E\in\mathscr{E}$, there is $F\in\mathscr{F}$ such that $E\subset F$, then $\mathscr{E}$ is said to be finer than $\mathscr{F}$ (denoted as $\mathscr{E}<\mathscr{F}$).

2. Comparison of various decompositions

It is possible to associate two types of Bishop decompositions for a complex function space [6]. We associate three different Bishop type decompositions with a vector function space and two types of Šilov decompositions.

Let V be a vector function space on X. Define

$$M(V)=\{f\in C(X;B)\colon fg\in V \text{ for all } g\in V\}$$

and

$$N(V)=\{f\in C(X)\colon (f\otimes e)g\in V \text{ for all } g\in V\}.$$

Then $M(V)$ and $N(V)$ are closed subalgebras of $C(X;B)$ and $C(X)$ respectively, $M(V)\subset V$ and $N(V)\hat{\otimes}B\subset V$. If V is an algebra, then $M(V)=V$.

DEFINITION 2.1.

A subset K of X is said to be a set of antisymmetry in (B)-sense for V, if $f\in M(V)$ and $\phi o f|_K$ real-valued for each $\phi\in m(B)$ implies that $f|_K$ is constant. The collection of all maximal (B)-antisymmetric sets for V forms a decomposition of X and is denoted by $\mathscr{K}(V)$.

A set of constancy of $(M(V))_{\mathbb{R}}$ is called an (S)-Šilov set for V, where $(M(V))_{\mathbb{R}}=\{f\in M(V)\colon \phi o f \text{ is real valued for all } \phi\in m(B)\}$. The collection of all maximal (S)-Šilov sets forms a decomposition of X and is denoted by $\mathscr{F}(V)$.

$N(V)$ is a subalgebra of $C(X)$. The usual antisymmetric and Šilov decompositions for $N(V)$ are called the (E)-antisymmetric and (E)-Šilov decompositions of V and are denoted by $\mathscr{K}_E(V)$ and $\mathscr{F}_E(V)$ respectively. This generalizes the definition of Edwards [2].

The decompositions defined in [3] are called (FP)-antisymmetric and (FP)-Šilov decompositions respectively. They are denoted by $\mathscr{K}_{\mathrm{FP}}(V)$ and $\mathscr{F}_{\mathrm{FP}}(V)$. Recall that an (FP)-antisymmetric set for V is a subset K of X for which $f\in N(V|_K)$ for f real-valued implies f is constant. It is easy to see that $\mathscr{F}_{\mathrm{FP}}(V)=\mathscr{F}_E(V)$.

Now onwards we omit the letter V in the paranthesis for these decompositions i.e., we shall write $\mathscr{K}$ in place of $\mathscr{K}(V)$ etc.

Remarks 2.2. (i) If $B=\mathbb{C}$, then $M(V)=N(V)$, $m(B)$ consists of only one point and hence $\mathscr{K}_E=\mathscr{K}$ and $\mathscr{F}_E=\mathscr{F}$. Further if V is an algebra, then $\mathscr{K}_{\mathrm{FP}}=\mathscr{K}_E(=\mathscr{K})$ and $\mathscr{F}_E=\mathscr{F}$ are the usual decompositions.
(ii) It is clear from the definitions that $\mathscr{K}<\mathscr{F}$ and $\mathscr{K}_E<\mathscr{F}_E$. Also, since $N(V)|_K\subset N(V|_K)$, we get $\mathscr{K}_{\mathrm{FP}}<\mathscr{K}_E$.

PROPOSITION 2.3.

Let V be a vector function space on X. Then (i) $\mathscr{K}<\mathscr{K}_E$ *and* $\mathscr{F}<\mathscr{F}_E$. (ii) *If V is an algebra, then $\mathscr{K}<\mathscr{K}_{\mathrm{FP}}$. Consequently, $\mathscr{K}<\mathscr{K}_{\mathrm{FP}}<\mathscr{K}_E$.*

Proof. (i) Let $F\in\mathscr{K}$ and $g\in N(V)$, $g|_F$ real-valued. Then $g\otimes e\in M(V)$ and $\phi\mathrm{o}(g\otimes e)|_F=g|_F$ is real-valued for every $\phi\in m(B)$. Hence $g\otimes e|_F$ is constant as $F\in\mathscr{K}$. Therefore $g|_F$ is constant. So F is an (E)-antisymmetric set for V or $\mathscr{K}<\mathscr{K}_E$. Similarly it can be shown that $\mathscr{F}<\mathscr{F}_E$.
(ii) Let V be an algebra, $K\in\mathscr{K}$ and $f\in N(V|_K)$ be real-valued. Since $f\otimes e\in V|_K$, there is $g\in V$ such that $g|_K=f\otimes e$. Then $\phi\mathrm{o}g|_K=f$ is real-valued for each $\phi\in m(B)$. So $g|_K$ is constant and hence f is constant. Thus K is an (FP)-antisymmetric set or $\mathscr{K}<\mathscr{K}_{\mathrm{FP}}$.

As we shall see in the last section, the decompositions $\mathscr{K}$, $\mathscr{K}_{\mathrm{FP}}$ and $\mathscr{K}_E$ may all be different. It is well-known that even for a function algebra, the Bishop and Šilov decompositions may not coincide in general [4, p. 60]. Similarly both the Bishop and the Šilov decompositions are all different for a complex function space. However under certain conditions some of these decompositions may coincide [6]. It is natural to discuss such problems for all these five decompositions for a vector function space.

For a vector function space V and $\phi\in m(B)$, let $V_\phi=\{\phi\mathrm{o}f:f\in V\}$. Then V_ϕ is a subspace of $C(X)$ and is a subalgebra, if V is an algebra. Further, $N(V)\subset(M(V))_\phi\subset V_\phi$ for every $\phi\in m(B)$. It is easy to verify that for $V=A\hat{\otimes}B$ (Ex. 1.2 (i)), $V_\phi=A$ for each $\phi\in m(B)$. Also it is not difficult to check that, in this case, $N(V)=N(A)=(M(V))_\phi$ for each $\phi\in m(B)$. Further, if V is as in Example 1.2 (ii), then also it can be proved that $N(V)=N(A)$.

PROPOSITION 2.4.

Let V be a vector function space on X. If $N(V)=V_\phi$ for some $\phi\in m(B)$, then $\mathscr{K}_{\mathrm{FP}}=\mathscr{K}_E$.

Proof. It suffices to show that $N(V)|_K=N(V|_K)$ for every subset K of X. Obviously, $N(V)|_K\subset N(V|_K)$. Now let $f\in N(V|_K)$. Choose $g\in V$ such that $f\otimes e=g|_K$. Fix $\phi\in m(B)$ so that $N(V)=V_\phi$. Now $\phi\mathrm{o}g\in V_\phi=N(V)$. Also $\phi\mathrm{o}g|_K=\phi\mathrm{o}(g|_K)=f$. Therefore, $f\in N(V)|_K$. Thus $N(V|_K)\subset N(V)|_K$ which completes the proof.

Remark 2.5. The condition "$N(V)=V_\phi$ for some $\phi\in m(B)$" is not a necessary condition for $\mathscr{K}_{\mathrm{FP}}=\mathscr{K}_E$ (Ex. 4.2(a)). Also it cannot be replaced by the weaker condition "$N(V)=(M(V))_\phi$ for some $\phi\in m(B)$" (Example 4.2(b)).

PROPOSITION 2.6.

Let V be a vector function space on X and B be semisimple. If $N(V)=(M(V))_\phi$ for every $\phi\in m(B)$, then $\mathscr{K}=\mathscr{K}_E$ and $\mathscr{F}=\mathscr{F}_E$.

Proof. By Proposition 2.3 (i), $\mathscr{K} < \mathscr{K}_E$. Let $K \in \mathscr{K}_E$ and $f \in M(V)$ such that $\phi o f|_K$ is real-valued for all $\phi \in m(B)$. Since $N(V) = (M(V))_\phi$ for every $\phi \in m(B)$, $\phi o f \in N(V)$ for all $\phi \in m(B)$. Hence $\phi o f|_K$ is constant for each ϕ. Since B is semisimple, we get $f|_K$ is constant. Thus K is a (B)-antisymmetric set for V and hence $\mathscr{K}_E < \mathscr{K}$. Similarly, it can be shown that $\mathscr{F} = \mathscr{F}_E$.

COROLLARY 2.7.

Let A be a complex function space on X and B be semisimple. Then $\mathscr{K}(A \hat{\otimes} B) = \mathscr{K}_E(A \hat{\otimes} B)$ and $\mathscr{F}(A \hat{\otimes} B) = \mathscr{F}_E(A \hat{\otimes} B)$.

Proof. As we have noted earlier, for $V = A \hat{\otimes} B$,

$$N(V) = (M(V))_\phi \text{ for each } \phi \in m(B).$$

Remark 2.8. The converse of Proposition 2.6 does not hold (Example 4.3).

For complex function spaces and algebras, the various decompositions associated with them have a very interesting and useful property, the (D)-property [5]. Hayashi [5] has defined and discussed some such properties of the decomposition of X for closed subalgebras of $C(X)$. These properties have been discussed for complex function spaces in [7]. These properties can be defined for vector function spaces in a similar fashion.

Remarks 2.9. (i) If V is a vector function space and $\mathscr{E}$ is a decomposition of X, then (GA)-property $\Rightarrow$ (S)-property $\Rightarrow$ (D)-property of $\mathscr{E}$ for V. Also, if $\mathscr{E}_1 < \mathscr{E}_2$ and $\mathscr{E}_1$ has any one of these properties for V, $\mathscr{E}_2$ also has the same property.
(ii) $\mathscr{K}_{FP}$ has the (GA)-property for V [3]. So, by (i) above, each of $\mathscr{K}_{FP}$, $\mathscr{K}_E$ and $\mathscr{F}_E$ has all these properties. But, in general, $\mathscr{K}$ and $\mathscr{F}$ do not have even the (D)-property for V (Remark 3.7 and Example 4.1(b)).
(iii) The results Theorem 2.1.7 of [6] and Theorem 3.1 of [7] can be generalized for a vector function space by similar technique.

For a complex function space A, we have proved that if $\mathscr{F}(A)$ is finite then $\mathscr{K}_{FP}(A) = \mathscr{K}_E(A) = \mathscr{F}(A)$ [6, Theorem 2.1.10]. Hence, from the definitions, it follows that if $\mathscr{F}_E(V)$ is finite, then $\mathscr{K}_E(V) = \mathscr{F}_E(V)$. Also with a slight modification in the proof, it can be shown that $\mathscr{K}_{FP}(V) = \mathscr{K}_E(V)$. In proving the above mentioned result, we have used the fact that $\mathscr{F}(A)$ has the (D)-property for A. For a vector function space V, a parallel result for $\mathscr{K}(V)$ and $\mathscr{F}(V)$ can be proved in spite of the absence of the (D)-property.

PROPOSITION 2.10.

Let V be a vector function space on X. If $\mathscr{F}(V)$ is finite, then $\mathscr{K}(V) = \mathscr{F}(V)$.

Proof. Let $\mathscr{F}(V) = \{F_1, F_2, \ldots, F_n\}$. For $F_1 \neq F_j$, there exists f_{1j} in $(M(V))_R$ such that $f_{1j} = e$ on F_1 and $f_{1j} = 0$ on F_j. Take $f_1 = f_{12} \cdots f_{1n}$. Then $f_1 \in (M(V))_R$, $f_1 = e$ on F_1 and $f_1 = 0$ on F_j for all $j \neq 1$. Similarly, for each $i = 2, 3, \ldots, n$; there exists $f_i \in (M(V))_R$ such that $f_i = e$ on F_i and $f_i = 0$ on F_j for all $j \neq i$.

Fix F_k in $\mathscr{F}(V)$. Let $g \in M(V)$ and $(\phi o g)|_{F_k}$ be real-valued for all $\phi \in m(B)$. Then $g f_k \in M(V)$. Also, $\phi o(g f_k)|_{F_k} = (\phi o g)|_{F_k} (\phi o f_k)|_{F_k} = (\phi o g)|_{F_k}$ and $\phi o(g f_k)|_{F_i} = (\phi o g)|_{F_i} (\phi o f_k)|_{F_i} = 0$ for $i \neq k$. Thus $\phi o(g f_k)$ is real-valued for all $\phi \in m(B)$. Therefore

$gf_k \in (M(V))_{\mathbb{R}}$ and so, $(gf_k)|_{F_k}$ is constant. Hence $g|_{F_k}$ is constant and F_k is a (B)-antisymmetric set for V. Thus $\mathscr{F}(V) < \mathscr{K}(V)$. Since $\mathscr{K}(V) < \mathscr{F}(V)$, we get $\mathscr{K}(V) = \mathscr{F}(V)$.

We shall give some examples in the last section to show that neither of the decompositions $\mathscr{K}$, $\mathscr{K}_{\mathrm{FP}}$ or $\mathscr{K}_E$ determine the other. Before that we shall obtain all these decompositions for two special classes of vector function spaces, defined in § 1.

3. Decompositions for special vector function spaces

Let A be a complex function space on X and B be a semisimple Banach algebra. It is natural to expect that the decompositions of $A \hat{\otimes} B$ and A are related. We show that they are, in fact, equal.

PROPOSITION 3.1.

$\mathscr{K}(A \hat{\otimes} B) = \mathscr{K}_E(A \hat{\otimes} B) = \mathscr{K}_E(A)$ and $\mathscr{F}(A \hat{\otimes} B) = \mathscr{F}_E(A \hat{\otimes} B) = \mathscr{F}_E(A)$.

Proof. The first part of both the equalities follows from Corollary 2.7. Also $N(A \hat{\otimes} B) = N(A)$ as we have noted earlier. Therefore, $\mathscr{K}_E(A \hat{\otimes} B) = \mathscr{K}_E(A)$ and $\mathscr{F}_E(A \hat{\otimes} B) = \mathscr{F}_E(A)$.

PROPOSITION 3.2.

For a closed subset K of X, $N(A \hat{\otimes} B|_K) \subset N(A|_K)$. Consequently $\mathscr{K}_{\mathrm{FP}}(A) < \mathscr{K}_{\mathrm{FP}}(A \hat{\otimes} B)$.

Proof. Let $f \in N(A \hat{\otimes} B|_K)$ and $g|_K \in A|_K$ with $g \in A$. Then $(f \otimes e)\ (g \otimes e)|_K \in A \hat{\otimes} B|_K$. Hence there exists $h \in A \hat{\otimes} B$ such that $h|_K = (f \otimes e)(g \otimes e)|_K = (fg \otimes e)|_K$. Now, for $\phi \in m(B)$, $\phi \circ h \in A$ and hence $fg|_K = \phi \circ (fg \otimes e)|_K = \phi \circ h|_K \in A|_K$. Thus $f \in N(A|_K)$. Therefore $N(A \hat{\otimes} B|_K) \subset N(A|_K)$.

To prove, $\mathscr{K}_{\mathrm{FP}}(A) = \mathscr{K}_{\mathrm{FP}}(A \hat{\otimes} B)$, we shall show that $\mathscr{K}_{\mathrm{FP}}(A)$ has the (S)-property for $A \hat{\otimes} B$. In fact, we prove a stronger result.

Theorem 3.3. *If A is a complex function space on X, then $\mathscr{K}_{\mathrm{FP}}(A)$ has the (GA)-property for $A \hat{\otimes} B$ (i.e., for each $\mu \in (b(A \hat{\otimes} B)^{\perp})^e$, supp $\mu \subset K$ for some $K \in \mathscr{K}_{\mathrm{FP}}(A)$, where $(A \hat{\otimes} B)^{\perp}$ is the annihilator of $A \hat{\otimes} B$ in the dual of $C(X; B)$) and hence $\mathscr{K}_{\mathrm{FP}}(A)$ has the (S)-property for $A \hat{\otimes} B$.*

Proof. It is known that the dual of $C(X, B)$ is $M(X; B^*)$, the set of all weak*-regular, B^*-valued measures of bounded variation, where B^* denotes the dual of B [8]. Let $\mu \in (b(A \hat{\otimes} B)^{\perp})^e$, the set of extreme points of the unit ball of $(A \hat{\otimes} B)^{\perp}$ and let $S = \operatorname{supp} \mu$. It is enough to show that S is an (FP)-antisymmetric set for A.

Let $f \in N(A|_S)$ and f be real-valued. Assume that $0 \leqslant f \leqslant \frac{1}{2}$ and let $\eta = f\mu$. Clearly $\eta \in M(X; B^*)$. Let $g = \Sigma_i g_i \otimes b_i \in A \hat{\otimes} B$ with $g_i \in A$, $b_i \in B$ for each i. Since $f \in N(A|_S)$, $fg_i|_S \in A|_S$ for each i. Therefore $fg|_S = \Sigma_1 fg_i \otimes b_i|_S \in A \hat{\otimes} B|_S$. Hence $\int_X g\,\mathrm{d}\eta = \int_S gf\,\mathrm{d}\mu = 0$ as $\mu \in (A \hat{\otimes} B)^{\perp}$. Thus $\eta \in (A \hat{\otimes} B)^{\perp}$. Now, it is easy to check that $\|f\mu\| + \|(1-f)\mu\| = 1$. Take $\eta_1 = \dfrac{f\mu}{\|f\mu\|}$ and $\eta_2 = \dfrac{(1-f)\mu}{\|(1-f)\mu\|}$. Then η_1, $\eta_2 \in b(A \hat{\otimes} B)^{\perp}$ and $\mu = \|f\mu\| \eta_1 +$

$\|(1-f)\mu\|\eta_2$. Since μ is an extreme point, we must have $\mu=\eta_1=\eta_2$. Therefore $f|_S$ is constant on S and S is an (FP)-antisymmetric set for A.

PROPOSITION 3.4.

For a complex function space A on X, $\mathscr{K}_{\mathrm{FP}}(A)=\mathscr{K}_{\mathrm{FP}}(A\hat{\otimes}B)$.

Proof. We have shown that $\mathscr{K}_{\mathrm{FP}}(A)<\mathscr{K}_{\mathrm{FP}}(A\hat{\otimes}B)$ in Proposition 3.2. Now $\mathscr{K}_{\mathrm{FP}}(A)$ is Hausdorff decomposable and has the (S)-property for $A\hat{\otimes}B$ by the above theorem. Hence by the Remark 2.9 (iii), $\mathscr{K}_{\mathrm{FP}}(A\hat{\otimes}B)=\mathscr{K}_{\mathrm{FP}}(A)$.

Next we consider another interesting class of vector function spaces. Let A be a complex function space on X and B be a function algebra on Y. Fix a point $y\in Y$ and define $V=\{f\in C(X;B):\phi_y\mathrm{o}f\in A\}$, where $\phi_y:B\to\mathbb{C}$ is defined by $\phi_y(g)=g(y)$. Then V is a vector function space on X. We shall discuss the decompositions of V with different conditions on B. First we make some simple observations.

Remarks 3.5 (i) As we have noted earlier, $N(V)=N(A)$ in this case also and hence $\mathscr{K}_E(V)=\mathscr{K}_E(A)$ and $\mathscr{F}_E(V)=\mathscr{F}_E(A)$.
(ii) Exactly as in Proposition 3.2, it can be proved that $\mathscr{K}_{\mathrm{FP}}(A)<\mathscr{K}_{\mathrm{FP}}(V)$. In general, the equality may not hold. However, if A is an algebra, then $\mathscr{K}_{\mathrm{FP}}(A)=\mathscr{K}_E(A)$ and hence we get $\mathscr{K}_{\mathrm{FP}}(A)=\mathscr{K}_{\mathrm{FP}}(V)$, as $\mathscr{K}_{\mathrm{FP}}(V)<\mathscr{K}_E(V)=\mathscr{K}_E(A)=\mathscr{K}_{\mathrm{FP}}(A)$. This result also follows from Proposition 2.4 and (i) above as $N(A)=A=V_{\phi_y}$ in this case.

Thus the decompositions $\mathscr{K}_{\mathrm{FP}}$, $\mathscr{K}_E$ and $\mathscr{F}_E$ for V are related with the corresponding decompositions of A. But the decomposition $\mathscr{K}$ for V depends heavily upon the usual Bishop decomposition $\mathscr{K}(B)$ of B as we shall see.

PROPOSITION 3.6.

(i) *If $\mathscr{K}(B)=\{Y\}$, then $\mathscr{K}(V)=\mathscr{K}_E(V)$* and *$\mathscr{F}(V)=\mathscr{F}_E(V)$.*
(ii) *If $\mathscr{K}(B)$ contains more than one element, then $\mathscr{K}(V)=\{\{x\}:x\in X\}$.*

Proof. (i) Since $\mathscr{K}(V)<\mathscr{K}_E(V)=\mathscr{K}_E(A)$ always holds, we need to show that $\mathscr{K}_E(V)<\mathscr{K}(V)$ in this case. Let $K\in\mathscr{K}_E(V)$, $f\in M(V)$ so that $\phi\mathrm{o}f$ is real-valued on K for each $\phi\in m(B)$. Then $f(x)\in B$ and is a real-valued function for each $x\in K$, i.e., $f(x)\in B_{\mathbb{R}}$. Since $\mathscr{K}(B)=\{Y\}$, $f(x)$ is constant on Y, say $f(x)=\alpha_x$ for each $x\in K$.

Next we show that $\phi_y\mathrm{o}f\in N(A)$. Let $g\in A$. Then $g\otimes e\in V$ and hence $f(g\otimes e)\in V$ as $f\in M(V)$. Therefore $\phi_y\mathrm{o}(f(g\otimes e))=(\phi_y\mathrm{o}f)g\in A$. Thus $\phi_y\mathrm{o}f\in N(A)$. Now $K\in\mathscr{K}_E(V)=\mathscr{K}_E(A)$ and $\phi_y\mathrm{o}f$ is real-valued on K. Hence $\phi_y\mathrm{o}f|_K$ is constant. Thus, for $x_1,x_2\in K$, $\alpha_{x_1}=(f(x_1))(y)=(\phi_y\mathrm{o}f)(x_1)=(\phi_y\mathrm{o}f)(x_2)=(f(x_2))(y)=\alpha_{x_2}$, i.e., $f(x_1)=f(x_2)$ and hence f is constant on K. Therefore, $\mathscr{K}_E(V)<\mathscr{K}(V)$.

Similarly, we can show that $\mathscr{F}(V)=\mathscr{F}_E(V)$.
(ii) Let $x_1,x_2\in X$, $x_1\neq x_2$. Then there exists $f\in C_{\mathbb{R}}(X)$ such that $f(x_1)\neq f(x_2)$. Since $\mathscr{K}(B)$ contains more than one elements, there is $g\in B_{\mathbb{R}}$ such that g is not a constant function and $g(y)=0$. Let $h=f\otimes g$. Then $h\in C(X;B)$ and $\phi_y\mathrm{o}h=f.g(y)=0\in A$. Therefore $h\in V$. To show that $h\in M(V)$, let $h'\in V$. Then $\phi_y\mathrm{o}(hh')=(\phi_y\mathrm{o}h)(\phi_y\mathrm{o}h')=0\in A$ and hence $hh'\in V$. Therefore, $h\in M(V)$. Also $\phi\mathrm{o}h$ is real-valued for all $\phi\in m(B)$. But $h(x_1)=f(x_1)g\neq f(x_2)g=h(x_2)$. Thus $\mathscr{F}(V)=\{\{x\}:x\in X\}$ and hence $\mathscr{K}(V)=\{\{x\}:x\in X\}$.

Remark 3.7. The decomposition $\{\{x\}:x\in X\}$ has the (D)-property for a vector function space V if and only if $V=C(X;B)$. So in the above result if $A \subsetneqq C(X)$ and B as in (ii), then $\mathscr{K}(V)$ does not have the (D)-property for V.

4. Examples

Here we shall give various examples of vector function spaces as promised in earlier sections. We mainly consider different cases of the two categories of vector function spaces discussed in the previous section.

We begin with examples wherein at least two of the three decompositions $\mathscr{K}$, $\mathscr{K}_{\mathrm{FP}}$ and $\mathscr{K}_E$ are different.

Examples 4.1. (a) Let A be a complex function space on X with $\mathscr{K}_{\mathrm{FP}}(A)\neq\mathscr{K}_E(A)$ [7, Example 4.2]. Let $V=A\hat{\otimes}B$. Then, by Propositions 3.1 and 3.4, $\mathscr{K}_{\mathrm{FP}}(V)\neq\mathscr{K}_E(V)=\mathscr{K}(V)$.
(b) Let A be a proper function space on X and $B=C\,[0,1]$. Take $V=\{f\in C(X;B):\phi_y\circ f\in A$ for a fixed $y\in[0,1]\}$. Then as we have seen in the previous section, $\mathscr{K}_E(V)=\mathscr{K}_E(A)$ and $\mathscr{K}(V)=\{\{x\}:x\in X\}$. Thus $\mathscr{K}(V)\neq\mathscr{K}_E(V)=\mathscr{K}_E(A)$ as A is proper.

The next two examples show that the condition in Proposition 2.4 is not necessary for $\mathscr{K}_{\mathrm{FP}}=\mathscr{K}_E$, and the condition therein cannot be weakened.

Examples 4.2. (a) Let A be a complex function space on X with $\mathscr{K}_{\mathrm{FP}}(A)=\mathscr{K}_E(A)$ and let $V=A\hat{\otimes}B$. Then $\mathscr{K}_E(V)=\mathscr{K}_E(A)=\mathscr{K}_{\mathrm{FP}}(A)=\mathscr{K}_{\mathrm{FP}}(V)$ (Propositions 3.1 and 3.4). But $N(V)=N(A)\subsetneqq A=V_\phi$ for every $\phi\in m(B)$.
(b) Again let A be a complex function space on X such that $\mathscr{K}_{\mathrm{FP}}(A)\neq\mathscr{K}_E(A)$ and $V=A\hat{\otimes}B$. Then, it can be checked that $M(V)\subset\{f\in C(X;B):\phi\circ f\in N(A)$ for all $\phi\in m(B)\}$. Since $N(A)=N(V)$ in this case, we get $(M(V))_\phi\subset N(A)=N(V)$, for each $\phi\in m(B)$. Since $N(V)\subset(M(V))_\phi$ always holds, we get $N(V)=(M(V))_\phi$ for each $\phi\in m(B)$. However, $\mathscr{K}_{\mathrm{FP}}(V)\neq\mathscr{K}_E(V)$.

The following example shows that the converse of the Proposition 2.6 does not hold.

Example 4.3. Let A be a function algebra on X and $B=A(D)$, the disc algebra on the unit disc D. Let $V=\{f\in C(X;B):\phi_y\circ f\in A$ for a fixed $y\in D\}$.
Then, by Proposition 3.6 (i), $\mathscr{K}(V)=\mathscr{K}_E(V)$. Since V is an algebra, $V=M(V)$. If $\phi\neq\phi_y$ then it can be checked that $V_\phi=C(X)$. But $N(V)=N(A)=A\subsetneqq C(X)$, and hence $N(V)\neq V_\phi$ wherever $\phi\neq\phi_y$.

Finally, we give examples to show that neither of the decompositions $\mathscr{K}$, $\mathscr{K}_{\mathrm{FP}}$ or $\mathscr{K}_E$ determine the other.

Example 4.4. For complex function spaces, it is known that neither $\mathscr{K}_{\mathrm{FP}}$ nor $\mathscr{K}_E$ determine the other [7, Example 4.4]. If we consider the vector function spaces obtained by taking the tensor product of these complex function spaces with B, then it follows that for vector function spaces also, neither $\mathscr{K}_{\mathrm{FP}}$ nor $\mathscr{K}_E$ determine the other.

Since $\mathscr{K}=\mathscr{K}_E$ for tensor products, it follows that neither $\mathscr{K}$ nor $\mathscr{K}_{\mathrm{FP}}$ determine the other.

Examples 4.5. (a) Let A_1 and A_2 be two function algebras with $\mathscr{K}(A_1) \neq \mathscr{K}(A_2)$ and $B = C[0,1]$. For $i = 1, 2$, let $V_i = \{f \in C(X; B): \phi_y o f \in A_i$ for a fixed $y \in [0,1]\}$. Then $\mathscr{K}(V_1) = \mathscr{K}(V_2) = \{\{x\}: x \in X\}$, but $\mathscr{K}_E(V_1) = \mathscr{K}(A_1) \neq \mathscr{K}(A_2) = \mathscr{K}_E(V_2)$. Thus $\mathscr{K}$ does not determine $\mathscr{K}_E$.
(b) Let A be a function algebra and $B = C[0,1]$. Let $V_1 = A \hat{\otimes} B$ and $V_2 = \{f \in C(X; B): \phi_y o f \in A$ for a fixed $y \in [0,1]\}$. Then $\mathscr{K}(V_1) = \mathscr{K}_E(V_1) = \mathscr{K}(A) = \mathscr{K}_E(V_2)$, but $\mathscr{K}(V_2) = \{\{x\}: x \in X\}$. Thus $\mathscr{K}_E$ does not determine $\mathscr{K}$.

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On M_2-graded hypergroups

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Abstract. In this paper, we introduce a notion called M_2-graded hypergroup, which extends the notion of hypergroup and is motivated by the example of a 'paragroup' in the context of the inclusion of a pair of II_1 factors. After discussing the example of the 'paragroup' we derive certain consequences of the definition and then prove that every finite irreducible M_2-graded hypergroup possesses a unique dimension function, in analogy with a result for hypergroups.

Keywords. M_2-graded hypergroups.

1. Introduction

We define M_2-graded hypergroups, a notion which is a natural generalization of the notion of hypergroups introduced in [S]. This is motivated by an example in the context of the inclusion of a pair of II_1 factors $N \subseteq M$. In fact, our axioms in the definition of an M_2-graded hypergroup describe the algebraic structure possessed by the collection of all isomorphism classes of irreducible N–N, N–M, M–N, and M–M bifinite bimodules with respect to tensor products and contragredients. We discuss the above example and then derive certain consequences of the definition. Finally we extend the notion of the dimension function of a hypergroup to M_2-graded hypergroups and prove that every finite irreducible M_2-graded hypergroup admits a unique dimension function.

DEFINITION 1

A discrete M_2-graded hypergroup is a set $\mathscr{G} = \bigcup_{1 \leqslant i,j \leqslant 2} \mathscr{G}_{ij}$ (where $\cup$ denotes disjoint union) with a map from $\mathscr{G} \times \mathscr{G}$ to $Z^+\mathscr{G} (Z^+ = \{0,1,2,\ldots\})$ denoted by

$$(\alpha,\beta) \mapsto \alpha \otimes \beta = \sum_{\gamma \in \mathscr{G}} \langle \alpha \otimes \beta, \gamma \rangle \gamma,$$

satisfying the following conditions:

(i) $\langle \alpha \otimes \beta, \gamma \rangle \neq 0$ for only finitely many γ, for all $\alpha, \beta \in \mathscr{G}$;
(ii) $\mathscr{G}_{ij} \otimes \mathscr{G}_{kl} \subseteq \delta_{jk} Z^+ \mathscr{G}_{il}$ and $Z\mathscr{G}$ is an associative algebra;
(iii) there exist $1_{\mathscr{G}_{ii}} \in \mathscr{G}_{ii}$, $i = 1, 2$, such that,

$$1_{\mathscr{G}_{ii}} \otimes \alpha = \alpha \text{ for all } \alpha \in \mathscr{G}_{ij}, \quad 1 \leqslant i,j \leqslant 2,$$

$$\alpha \otimes 1_{\mathscr{G}_{ii}} = \alpha \text{ for all } \alpha \in \mathscr{G}_{ji}, \quad 1 \leqslant i,j \leqslant 2;$$

(iv) there exists a self map of $\mathscr{G}$ denoted by $\alpha\mapsto\bar{\alpha}$ such that

$$\alpha\in\mathscr{G}_{ij} \text{ implies } \bar{\alpha}\in\mathscr{G}_{ji} (\text{i.e., } \bar{\mathscr{G}}_{ij}\subseteq\mathscr{G}_{ji})$$

and

$$\langle\alpha\otimes\beta,\gamma\rangle=\langle\bar{\alpha}\otimes\gamma,\beta\rangle, \text{ for all } \alpha,\beta\in\mathscr{G}.$$

Remark 2. (a) If $\mathscr{G}_{12}=\mathscr{G}_{21}=\mathscr{G}_{22}=\emptyset$, an M_2-graded hypergroup reduces to a hypergroup as in [S].

(b) The above definition is motivated by the following example:

Let $\mathscr{G}_{11}=\mathscr{G}(N)$, $\mathscr{G}_{12}=\mathscr{G}(N,M)$, $\mathscr{G}_{21}=\mathscr{G}(M,N)$ and $\mathscr{G}_{22}=\mathscr{G}(M)$; where N and M are II_1 factors and $\mathscr{G}(P,Q)$ represents the isomorphism classes of irreducible left **P** finite, right **Q** finite **P**–**Q** bimodules with the convention that $\mathscr{G}(P,P)=\mathscr{G}(P)$.

Here $\mathscr{G}_{11}$ and $\mathscr{G}_{22}$ are hypergroups, $\overline{\mathscr{G}_{12}}=\mathscr{G}_{21}$, $\mathscr{G}_{11}$ and $\mathscr{G}^{op}_{22}$ (the opposite hypergroup of $\mathscr{G}_{22}$, i.e., there exists an anti-isomorphism $\alpha\mapsto\alpha^0$ from $\mathscr{G}$ to $\mathscr{G}^{op}$, such that $\langle\alpha^0\otimes\beta^0,\gamma^0\rangle=\langle\beta\otimes\alpha,\gamma\rangle$) act on $\mathscr{G}_{12}$ (equivalently $\mathscr{G}_{22}$ and $\mathscr{G}^{op}_{11}$ act on $\mathscr{G}_{21}$) in the sense of [SV].

(c) Objects with similar structure, with the name "fusion algebra", were discussed in [O1] and in [EK], we were unaware of the former preprint, while we received a copy of the latter only after this work was completed. The definition that we give seems to be the most natural one in the context of bifinite bimodules over II_1 factor-subfactor pairs.

Given an M_2-graded hypergroup $\mathscr{G}$ we can consider $\mathbb{C}\mathscr{G}$, the class of all finitely supported complex valued functions on $\mathscr{G}$. The space $\mathbb{C}\mathscr{G}$ has a canonical basis $\{f_\alpha,\alpha\in\mathscr{G}\}$, given by $f_\alpha(\beta)=\delta_{\alpha\beta}$ (the Kronecker delta); thus $f=\sum_{\alpha\in\mathscr{G}}f(\alpha)f_\alpha$ for all $f\in\mathbb{C}\mathscr{G}$. We can make $\mathbb{C}\mathscr{G}$ an algebra by defining a 'convolution product' on $\mathbb{C}\mathscr{G}$, given by $f_\alpha * f_\beta=\sum_{\gamma\in\mathscr{G}}\langle\alpha\otimes\beta,\gamma\rangle f_\gamma$ (i.e., $f_\alpha * f_\beta(\gamma)=\langle\alpha\otimes\beta,\gamma\rangle$). More generally, for f,g in $\mathbb{C}\mathscr{G}$,

$$f*g=\sum_{\alpha,\beta,\gamma\in\mathscr{G}}f(\alpha)g(\beta)\langle\alpha\otimes\beta,\gamma\rangle f_\gamma,$$

where $f=\sum_{\alpha\in\mathscr{G}}f(\alpha)f_\alpha$ and $g=\sum_{\beta\in\mathscr{G}}g(\beta)f_\beta$. The associativity of $Z\mathscr{G}$ ensures that $\mathbb{C}\mathscr{G}$ is an associative algebra with respect to the above product. We can make $\mathbb{C}\mathscr{G}$ an innerproduct space, which we denote by **V** by demanding that $\{f_\alpha;\alpha\in\mathscr{G}\}$ is a (necessarily maximal) orthonormal set of vectors.

Now we can consider the left regular representation of $\mathbb{C}\mathscr{G}$. For f in $\mathbb{C}\mathscr{G}$, define the left multiplication operator L_f on **V** by $L_f g=f*g$. Clearly $f\mapsto L_f$ is an algebra homomorphism of $\mathbb{C}\mathscr{G}$ into $L(V)$ which is unital—$f_{1_{\mathscr{G}_{11}}}+f_{1_{\mathscr{G}_{22}}}$ is the identity of $\mathbb{C}\mathscr{G}$ and $L_{f_{1_{\mathscr{G}_{11}}}}+L_{f_{1_{\mathscr{G}_{22}}}}$ is the identity of $L(V)$—and faithful. Further if we define $f^*(\alpha)=\overline{f(\bar{\alpha})}$-the complex conjugate of $f(\bar{\alpha})$—then the contragradient axiom ensures that, $\langle L_f g,h\rangle=\langle g,L_{f^*}h\rangle$ for f,g,h in $\mathbb{C}\mathscr{G}$.

PROPOSITION 3

The equation $\tau(f)=f(1_{\mathscr{G}_{11}})+f(1_{\mathscr{G}_{22}})$ *defines a faithful positive trace on* $\mathbb{C}\mathscr{G}$ *for any* M_2*-graded hypergroup* $\mathscr{G}$.

Proof.

Clearly,

$$
\begin{aligned}
\tau(f) &= \langle L_f f_{1_{\mathscr{G}_{11}}}, f_{1_{\mathscr{G}_{11}}}\rangle + \langle L_f f_{1_{\mathscr{G}_{22}}}, f_{1_{\mathscr{G}_{22}}}\rangle. \\
\tau(f^* * f) &= \langle L_{f^* * f} f_{1_{\mathscr{G}_{11}}}, f_{1_{\mathscr{G}_{11}}}\rangle + \langle L_{f^* * f} f_{1_{\mathscr{G}_{22}}}, f_{1_{\mathscr{G}_{22}}}\rangle \\
&= \langle L_{f^*} L_f f_{1_{\mathscr{G}_{11}}}, f_{1_{\mathscr{G}_{11}}}\rangle + \langle L_{f^*} L_f f_{1_{\mathscr{G}_{22}}}, f_{1_{\mathscr{G}_{22}}}\rangle \\
&= \langle L_f f_{1_{\mathscr{G}_{11}}}, L_f f_{1_{\mathscr{G}_{11}}}\rangle + \langle L_f f_{1_{\mathscr{G}_{22}}}, L_f f_{1_{\mathscr{G}_{22}}}\rangle \\
&= \langle f * f_{1_{\mathscr{G}_{11}}}, f * f_{1_{\mathscr{G}_{11}}}\rangle + \langle f * f_{1_{\mathscr{G}_{22}}}, f * f_{1_{\mathscr{G}_{22}}}\rangle \\
&= \langle f * (f_{1_{\mathscr{G}_{11}}} + f_{1_{\mathscr{G}_{22}}}), f * (f_{1_{\mathscr{G}_{11}}} + f_{1_{\mathscr{G}_{22}}})\rangle \\
&= \langle f, f\rangle,
\end{aligned}
$$

(since $f * f_{1_{\mathscr{G}_{11}}} \in \bigoplus_{j=1}^{2} Z^+ \mathscr{G}_{ji}$ implies $\langle f * f_{1_{\mathscr{G}_{11}}}, f * f_{1_{\mathscr{G}_{22}}}\rangle = 0$) Therefore τ is faithful and positive.

$$
\begin{aligned}
\tau(f * g) &= \langle f * g, f_{1_{\mathscr{G}_{11}}}\rangle + \langle f * g, f_{1_{\mathscr{G}_{22}}}\rangle \\
&= \langle g, f^* * f_{1_{\mathscr{G}_{11}}}\rangle + \langle g, f^* * f_{1_{\mathscr{G}_{22}}}\rangle \\
&= \langle g, f^* * (f_{1_{\mathscr{G}_{11}}} + f_{1_{\mathscr{G}_{22}}})\rangle \\
&= \langle g, f^*\rangle \\
&= \sum_{\alpha \in \mathscr{G}} g(\alpha) f(\bar{\alpha}) \\
&= \sum_{\beta \in \mathscr{G}} f(\beta) g(\bar{\beta}), \text{ ($\alpha \mapsto \bar{\alpha}$ being an involution of $\mathscr{G}$, as shown below)} \\
&= \tau(g * f).
\end{aligned}
$$

Q.E.D.

Hence τ is a trace.

Now we will discuss some consequences of the definition.

PROPOSITION 4

Let $\mathscr{G} = \bigcup_{1 \leqslant i,j \leqslant 2} \mathscr{G}_{ij}$ be an M_2-graded hypergroup. Then,

(i) *$\mathscr{G}_{ii}$ are hypergroups, $i = 1, 2$;*
(ii) *$1_{\mathscr{G}_{ii}}$ are unique, $i = 1, 2$;*
(iii) *$\alpha \mapsto \bar{\alpha}$ is an involution in $\mathscr{G}$;*
(iv) *$\langle \alpha \otimes \beta, \gamma\rangle = \langle \gamma \otimes \bar{\beta}, \alpha\rangle = \langle \bar{\beta} \otimes \bar{\alpha}, \bar{\gamma}\rangle$ (i.e., $\overline{\alpha \otimes \beta} = \bar{\beta} \otimes \bar{\alpha}$).*

Proof. (i) By axiom (ii) in the definition, we have, $\mathscr{G}_{ii} \otimes \mathscr{G}_{ii} \subseteq Z^+ \mathscr{G}_{ii}$.
Hence we have a map: $\mathscr{G}_{ii} \otimes \mathscr{G}_{ii} \otimes \mathscr{G}_{ii} \mapsto Z^+$ given by, $(\alpha, \beta, \gamma) \mapsto \langle \alpha \otimes \beta, \gamma\rangle$.
The identity of $\mathscr{G}_{ii}$ is $1_{\mathscr{G}_{ii}}$ by axiom (iii) and $\mathscr{G}_{ii}$ is closed under contragredients by axiom (iv). The associativity of $\mathscr{G}_{ii}$ follows from axiom (ii).

(ii) The uniqueness of $1_{\mathscr{G}_{ii}}$ follows from the fact that the identity is unique in a hypergroup.
(iii) We have,

$$\begin{aligned} 1 &= \langle \alpha_{ij} \otimes 1_{\mathscr{G}_{jj}}, \alpha_{ij} \rangle \\ &= \langle \overline{\alpha_{ij}} \otimes \alpha_{ij}, 1_{\mathscr{G}_{jj}} \rangle, \text{ by the contragredient axiom} \\ &= \langle \overline{\overline{\alpha_{ij}}} \otimes 1_{\mathscr{G}_{jj}}, \alpha_{ij} \rangle, \\ &= \delta_{\overline{\overline{\alpha_{ij}}}, \alpha_{ij}}, \text{ since } \overline{\overline{\alpha_{ij}}} \in \mathscr{G}_{ij} \end{aligned}$$

Hence $\overline{\overline{\alpha_{ij}}} = \alpha_{ij}$.
(iv)

$$\begin{aligned} \langle \alpha \otimes \beta, \gamma \rangle &= \langle f_\alpha * f_\beta, f_\gamma \rangle \\ &= \tau(f_{\bar{\gamma}} * f_\alpha * f_\beta) \\ &= \tau(f_\beta * f_{\bar{\gamma}} * f_\alpha) \\ &= \langle f_{\bar{\gamma}} * f_\alpha, f_{\bar{\beta}} \rangle \\ &= \langle \bar{\gamma} \otimes \alpha, \bar{\beta} \rangle \\ &= \langle \gamma \otimes \bar{\beta}, \alpha \rangle. \end{aligned}$$

Note that $\langle \alpha \otimes \beta, \gamma \rangle = \tau(f_\alpha * f_\beta * f_{\bar{\gamma}})$.

If g is such that $g(1_{\mathscr{G}_{11}}) + g(1_{\mathscr{G}_{22}}) \in \mathbb{R}$, then $\tau(g) = \tau(g^*)$, hence

$$\begin{aligned} \langle \alpha \otimes \beta, \gamma \rangle &= \tau(f_\alpha * f_\beta * f_{\bar{\gamma}}) \\ &= \tau(f_\gamma * f_{\bar{\beta}} * f_{\bar{\alpha}}). \\ &= \langle \bar{\beta} \otimes \bar{\alpha}, \bar{\gamma} \rangle. \end{aligned}$$

Q.E.D.

We recall from [SV] the definition of an action of a hypergroup on a set.

DEFINITION 5

An action of a hypergroup $\mathscr{G}_0$ on a set $\mathscr{G}_1$ is a mapping $\pi_1 : Z^+\mathscr{G}_0 \mapsto \mathrm{End}(Z^+\mathscr{G}_1)$, which is a homomorphism in the sense that it satisfies the follwing conditions.

(i) $\pi_1(1_{\mathscr{G}_0}) = id_{\mathrm{End}(Z^+\mathscr{G}_1)}$;

(ii) $\pi_1(\alpha)\pi_1(\beta) = \sum_{\gamma \in \mathscr{G}_0} \langle \alpha \otimes \beta, \gamma \rangle \pi_1(\gamma)$ for all α, $\beta \in \mathscr{G}_0$;

(iii) $\pi_1(\bar{\alpha}) = \pi_1(\alpha)'$ for all $\alpha \in \mathscr{G}_0$; where "′" denotes the matrix transpose (the endomorphisms of $Z^+\mathscr{G}_1$ being represented by matrices with respect to the basis given by $\mathscr{G}_1$).

Remark 6. (i) In the example above $\mathscr{G}(N)$ and $\mathscr{G}(M)$ are hypergroups and they act on $\mathscr{G}(N, M)$ and $\mathscr{G}(M, N)$, respectively, in the sense of Definition 5.
(ii) Being given an M_2-graded hypergroup is equivalent to being given hypergroups $\mathscr{G}_{11}$ and $\mathscr{G}_{22}$ with commuting actions π and ρ of $\mathscr{G}_{11}$ and $\mathscr{G}_{22}^{op}$ respectively on $\mathscr{G}_{12}$

satisfying,

$$\sum_{\kappa_{12}\in\mathscr{G}_{12}} \pi(\alpha_{11})(\kappa_{12},\alpha_{12})\rho(\alpha^0_{22})(\delta_{12},\kappa_{12}) = \sum_{\kappa_{12}\in\mathscr{G}_{12}} \pi(\alpha_{11})(\delta_{12},\kappa_{12})\rho(\alpha^0_{22})(\kappa_{12},\alpha_{12})$$

(iii) Recall the definition of a Z_2-graded hypergroup: it is just a hypergroup $\mathscr{G}$ of the form $\mathscr{G} = \mathscr{G}_0 \cup \mathscr{G}_1$, where $\mathscr{G}_0$ is a subhypergroup of $\mathscr{G}$; the elements of $\mathscr{G}_0$ are thought of as having degree zero and the elements of $\mathscr{G}_1$ are thought of as having degree one and it is assumed that $\langle\alpha\otimes\beta,\gamma\rangle = 0$ unless $\deg(\alpha)+\deg(\beta) = \deg(\gamma)(\mathrm{mod}\,2)$. In this case it is easy to see that $\mathscr{G}_0$ acts on $\mathscr{G}_1$ in a natural fashion. Clearly Z_2-graded hypergroups yield M_2-graded hypergroups; in particular, if $\mathscr{G}$ is a hypergroup, then $\mathscr{G}\times Z_2$ is a Z_2-graded hypergroup, hence an M_2-graded hypergroup.

PROPOSITION 7

Let $\mathscr{G}_0$ be a finite hypergroup. Then,

(1) $e = \frac{1}{\sum_{\alpha\in\mathscr{G}_0} d^2_\alpha} \sum_{\alpha\in\mathscr{G}_0} d_\alpha\, \alpha$ *is a self-adjoint projection in the centre of* $\mathbb{C}\mathscr{G}_0$, *where* $\alpha\mapsto d_\alpha$ *denotes the unique dimension function* (*cf.* [5]) *of* $\mathscr{G}_0$.

(2) *if* π *is an action of* $\mathscr{G}_0$ *on* $\mathscr{G}_1$, *then the following are equivalent:*
   (i) *for all* $\gamma,\kappa\in\mathscr{G}_1$ *there exists* $\alpha\in\mathscr{G}_0$ *such that* $\langle\alpha\otimes\kappa,\gamma\rangle > 0$;
   (ii) $\pi(e)(\gamma,\kappa) > 0$, *for all* γ,κ *in* $\mathscr{G}_1$;
   (iii) $\pi(e)$ *is a rank one projection.*

Proof. (1) Since $\alpha\mapsto\bar\alpha$ is an involution in $\mathscr{G}_0$ and $d_\alpha = d_{\bar\alpha}$, for all α in $\mathscr{G}_0$, $e^* = e$. Also, for all α in $\mathbb{C}\mathscr{G}_0$, e satisfies the relation $\alpha * e = e * \alpha = d_\alpha e$ because the column vector with the αth coordinate equal to d_α is the common Perron–Frobenius eigenvector of R_α and L_α corresponding to eigenvalue d_α, where R_α and L_α belonging to End $(\mathbb{C}\mathscr{G}_0)$ are defined as

$$R_\alpha(\gamma,\kappa) = \langle\kappa\otimes\alpha,\gamma\rangle \text{ and } L_\alpha(\gamma,\kappa) = \langle\alpha\otimes\kappa,\gamma\rangle.$$

Hence,

$$\begin{aligned} e^2 &= e * \frac{1}{\sum_{\alpha\in\mathscr{G}_0} d^2_\alpha} \sum_{\alpha\in\mathscr{G}_0} d_\alpha \alpha \\ &= \frac{1}{\sum_{\alpha\in\mathscr{G}_0} d^2_\alpha} \sum_{\alpha\in\mathscr{G}_0} d_\alpha e * \alpha \\ &= \frac{1}{\sum_{\alpha\in\mathscr{G}_0} d^2_\alpha} \sum_{\alpha\in\mathscr{G}_0} d_\alpha d_\alpha e \\ &= e. \end{aligned}$$

(2) Suppose (i) holds. Then,

$$\pi(e)(\gamma,\kappa) = \frac{1}{\sum_{\alpha\in\mathscr{G}_0} d^2_\alpha} \sum_{\alpha\in\mathscr{G}_0} d_\alpha \pi(\alpha)(\gamma,\kappa)$$

$$=\frac{1}{\sum_{\alpha\in\mathscr{G}_0} d_\alpha^2}\sum_{\alpha\in\mathscr{G}_0} d_\alpha\langle\alpha\otimes\kappa,\gamma\rangle,$$

which is greater than 0 by (i);

Now assume that (ii) holds.

i.e.,

$$\frac{1}{\sum_{\alpha\in\mathscr{G}_0} d_\alpha^2}\sum_{\alpha\in\mathscr{G}_0} d_\alpha\langle\alpha\otimes\kappa,\gamma\rangle>0,$$

for all $\gamma,\kappa\in\mathscr{G}_1$.

Hence, $\langle\alpha\otimes\kappa,\gamma\rangle>0$ for at least one α in $\mathscr{G}_0$.

If (ii) holds, then $\pi(e)$ is a matrix with strictly positive entries. Therefore by the Perron–Frobenius theorem there exists a positive eigenvalue of $\pi(e)$ with both geometric and algebraic multiplicity equal to one which is also the maximum of all eigenvalues of $\pi(e)$. But $\pi(e)$ is a projection so that its eigenvalues are either zero or one. Thus the maximum eigenvalue one is of geometric multiplicity one. Hence the range of $\pi(e)$ is one dimensional.

Thus $\pi(e)$ is a rank one projection.

Suppose (iii) holds.

Then, since $\pi(1_{\mathscr{G}_0})$ is the identity matrix in End $(Z^+\mathscr{G}_1)$ (as π is a homomorphism) $\pi(e)=\frac{1}{\sum_{\alpha\in\mathscr{G}_0} d_\alpha^2}\sum_{\alpha\in\mathscr{G}_0} d_\alpha\pi(\alpha)$ has strictly positive diagonal entries. Hence the rank one projection $\pi(e)$ has nonzero entries everywhere. **Q.E.D.**

DEFINITION 8

(a) An action π (of a finite hypergroup $\mathscr{G}_0$ on a set $\mathscr{G}_1$) is said to be irreducible if the conditions (i)–(iii) in Proposition 7(2) are satisfied.

(b) An M_2-graded hypergroup $\mathscr{G}$ is said to be irreducible if the (commuting) actions of $\mathscr{G}_{11}$ and $\mathscr{G}_{22}^{op}$ on $\mathscr{G}_{12}$ are both irreducible.

Remark 9. It can be seen that the M_2-graded hypergroup obtained from the 'paragroup' (cf. [O2]) of a subfactor is irreducible, (This differs from the one discussed in Remark 2(b) in that one only considers irreducible bimodules that feature in the 'tower of the basic construction'.)

DEFINITION 10

A dimension function for an M_2-graded hypergroup $\mathscr{G}$ is a map $d:\mathscr{G}\mapsto(0,\infty)$, satisfying the following conditions:

(i) $d(\alpha)=d(\bar{\alpha})$, for all α in $\mathscr{G}$;

(ii) $d(\alpha_{ij})d(\beta_{jk})=\sum_{\gamma_{ik}\in\mathscr{G}_{ik}}\langle\alpha_{ij}\otimes\beta_{jk},\gamma_{ik}\rangle d(\gamma_{ik})$, for all $i,j,k,=1,2$.

Remark 11. From the above definition it follows that, restricted to the hypergroups $\mathscr{G}_{ii}$, $i=1,2$, the function d is a dimension function for $\mathscr{G}_{ii}$.

Theorem 12. *Every finite irreducible M_2-graded hypergroup admits a unique dimension function.*

Proof. Let $\mathcal{G} = \bigcup_{1 \leqslant i,j \leqslant 2} \mathcal{G}_{ij}$ denote a finite irreducible M_2-graded hypergroup.

A linear operator T on V – the innerproduct space $\mathbb{C}\mathcal{G}$ with orthonormal basis $\{f_\alpha; \alpha \in \mathcal{G}\}$ – is identified with the matrix $(t_{\alpha\beta})$ where $t_{\alpha\beta} = \langle Tf_\beta, f_\alpha \rangle$. Thus L_α (resp., R_α) denotes the matrix whose rows and columns are indexed by $\mathcal{G}$ and whose entry in the position (β, γ) is

$$\langle \alpha \otimes \gamma, \beta \rangle \text{ (resp., } \langle \gamma \otimes \alpha, \beta \rangle)$$

Let π^i_j denote the action of $\mathcal{G}_{ii}$ on $\mathcal{G}_{ij}$, for $i,j = 1,2$.

Existence

On $\mathcal{G}_{ii}$ define d to be equal to the dimension function of $\mathcal{G}_{ii}$, for $i = 1,2$.

Let,

$$e_i = \frac{1}{\sum_{\alpha_{ii} \in \mathcal{G}_{ii}} d(\alpha_{ii})^2} \sum_{\alpha_{ii} \in \mathcal{G}_{ii}} d(\alpha_{ii}) \alpha_{ii}, \quad i = 1,2.$$

It is clear that $e_i \in \mathbb{R}^+ \mathcal{G}_{ii}$, $e_i^2 = e_i = e_i^*$ (the adjoint of e_i in $\mathbb{C}\mathcal{G}_{ii}$), $d(e_i) = 1$ (d on $\mathcal{G}_{ii}$ extends by linearity to a function from $\mathbb{C}\mathcal{G}_{ii}$ to $\mathbb{C}$ which is clearly multiplicative), $\alpha_{ii} e_i = e_i \alpha_{ii} = d(\alpha_{ii}) e_i$, and $\pi^i_j(e_i)$ is a projection in End $(\mathbb{C}\mathcal{G}_{ij})$.

Since $\mathcal{G}$ is irreducible π^i_j, $i,j = 1,2$, are irreducible so that the matrices corresponding to $\pi^i_j(e_i)$ have strictly positive entries. Let V_{ij} be the Perron–Frobenius eigenvector of $\pi^i_j(e_i)$. Thus V_{ij} is a (column) vector, in the range of $\pi^i_j(e_i)$, with strictly positive entries which is unique up to scaling by a positive constant. We assume that V_{ii} is normalized such that $V_{ii}(1_{\mathcal{G}_{ii}}) = 1$, so that it is the dimension function vector for $\mathcal{G}_{ii}$. For $i \neq j$, we shall choose a suitable normalization later. Thus $\pi^i_j(e_i) V_{ij} = V_{ij}$, $i,j = 1,2$ (the Perron–Frobenius eigenvalue of $\pi^i_j(e_i)$ is 1, since $\pi^i_j(e_i)$ is a projection).

Since V_{ij} belongs to $\mathbb{R}^+ \mathcal{G}_{ij}$ it is of the form,

$$V_{ij} = \sum_{\alpha_{ij} \in \mathcal{G}_{ij}} C(\alpha_{ij}) \alpha_{ij}, \quad \text{for } i,j = 1,2,$$

where $C(\alpha_{ij})$ are nonzero positive constants.

Consider $\rho^i_j(\alpha_{ii})$ belonging to End $(\mathbb{C}\mathcal{G}_{ji})$ given by.

$$\rho^i_j(\alpha_{ii})(\gamma_{ji}, \kappa_{ji}) = \langle \kappa_{ji} \otimes \alpha_{ii}, \gamma_{ji} \rangle, \quad i,j = 1,2.$$

Since in $\mathcal{G}$, $\langle \kappa_{ji} \otimes \alpha_{ii}, \gamma_{ji} \rangle = \langle \overline{\alpha_{ii}} \otimes \overline{\kappa_{ji}}, \overline{\gamma_{ji}} \rangle$,
we have,

$$\rho^i_j(\alpha_{ii})(\gamma_{ji}, \kappa_{ji}) = \pi^i_j(\overline{\alpha_{ii}})(\overline{\gamma_{ji}}, \overline{\kappa_{ji}}),$$

so that the matrix corresponding to $\rho^i_j(e_i)$ has strictly positive entries. Clearly, $\pi^i_j(e_i)$ commutes with $\rho^j_i(e_j)$, for $i,j = 1,2$. Therefore V_{ij} is an eigenvector, hence the Perron–Frobenius eigenvector of $\rho^j_i(e_j)$. But $\overline{V_{ji}}$ is the Perron–Frobenius eigenvector of $\rho^j_i(e_j)$. Hence we may assume that $\overline{V_{ji}} = V_{ij}$.

Assertion:

(i) $V_{ij}\otimes\alpha_{ji}=\varphi(\alpha_{ji})\,V_{ii}$, where $\varphi(\alpha_{ji})$ is a positive constant depending on α_{ji};
(ii) $V_{ii}\otimes\alpha_{ij}=\psi(\alpha_{ij})\,V_{ij}$, where $\psi(\alpha_{ij})$ is a positive constant depending on α_{ij};
(iii) $\varphi(\alpha_{ij})=C(\overline{\alpha_{ij}})$;
(iv) $\dfrac{\psi(\alpha_{ij})}{C(\alpha_{ij})}=k_{ij}$, a constant depending on (i,j), for $i,j=1,2$.

Proof of the Assertion: Note that for $i=j$ the above assertions are about the hypergroups $\mathcal{G}_{ii}$ and hence $\varphi=\psi=C=d$ and consequently $k_{ii}=1$, $i=1,2$.

Therefore in the remaining part of the proof of the assertion we assume that $i\neq j$.

(i) In $\mathbb{C}\mathcal{G}$ for $\alpha_{ij}\in\mathcal{G}_{ij}$, for $i,j=1,2$,

$$(\alpha_{ii}\otimes V_{ij})\otimes\alpha_{ji}=\alpha_{ii}\otimes(V_{ij}\otimes\alpha_{ji})$$
$$\text{i.e., } R_{\alpha_{ji}}(\pi_j^i(\alpha_{ii})\,V_{ij})=\pi_i^i(\alpha_{ii})(R_{\alpha_{ji}}\,V_{ij}),$$

Multiplying both sides by $d(\alpha_{ii})$ and summing over α_{ii} in $\mathcal{G}_{ii}$, we get

$$R_{\alpha_{ji}}(\pi_j^i(e_i)\,V_{ij})=\pi_i^i(e_i)(R_{\alpha_{ji}}\,V_{ij})$$
$$\text{i.e., } R_{\alpha_{ji}}\,V_{ij}=\pi_i^i(e_i)(R_{\alpha_{ji}}\,V_{ij})$$

since, $\sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii})\alpha_{ii}=\left(\sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii})^2\right)e_i$, and $\pi_i^i(e_i)\,V_{ii}=V_{ii}$. Therefore $R_{\alpha_{ji}}\,V_{ij}$ is in the range of $\pi_i^i(e_i)$. Further, $R_{\alpha_{ji}}$ is a matrix with non-negative entries, with non-zero rows and columns (since, π_j^i is irreducible) and V_{ij} has strictly postive entries. Hence $R_{\alpha_{ji}}\,V_{ij}$ is a positive scalar multiple of the Perron–Frobenius eigenvector for $\pi_i^i(e_i)$ which is V_{ii}.

(ii) Again in $\mathbb{C}\mathcal{G}$,

$$(\alpha_{ii}\otimes V_{ii})\otimes\alpha_{ij}=\alpha_{ii}\otimes(V_{ii}\otimes\alpha_{ij})$$
$$\text{i.e., } R_{\alpha_{ij}}\pi_i^i(\alpha_{ii})\,V_{ii}=\pi_j^i(\alpha_{ii})R_{\alpha_{ij}}\,V_{ii}.$$

Multiplying both sides by $d(\alpha_{ii})$ and summing over α_{ii} in $\mathcal{G}_{ii}$, we get

$$R_{\alpha_{ij}}\,V_{ii}=\pi_j^i(e_i)R_{\alpha_{ij}}\,V_{ii}$$

since,

$$\sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii})\alpha_{ii}=\left(\sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii})^2\right)e_i,\quad\text{and } \pi_i^i(e_i)\,V_{ii}=V_{ii}.$$

Hence $R_{\alpha_{ij}}\,V_{ii}$ is in the range of $\pi_j^i(e_i)$. Since $R_{\alpha_{ij}}$ is a matrix with non-negative entries, with non-zero rows and columns and V_{ii} is a vector with strictly positive entries, $R_{\alpha_{ij}}\,V_{ii}$ is a vector with strictly positive entries. Hence $R_{\alpha_{ij}}\,V_{ii}$ is a positive scalar multiple of the Perron–Frobenius eigenvector of $\pi_j^i(e_i)$ which is V_{ij}.

(iii) We have,

$$V_{ij}=\sum_{\alpha_{ij}\in\mathcal{G}_{ij}} C(\alpha_{ij})\alpha_{ij},$$
$$V_{ij}\otimes\alpha_{ji}=\varphi(\alpha_{ji})\,V_{ii}.$$

Since $C(1_{\mathscr{G}_{ji}}) = 1$, we see that

$$\begin{aligned}\varphi(\alpha_{ji}) &= \langle V_{ij} \otimes \alpha_{ji}, 1_{\mathscr{G}_{ii}} \rangle \\ &= \langle 1_{\mathscr{G}_{ii}} \otimes V_{ij}, \overline{\alpha_{ji}} \rangle \\ &= \langle V_{ij}, \overline{\alpha_{ji}} \rangle \\ &= C(\overline{\alpha_{ji}}).\end{aligned}$$

(iv) In $\mathbb{C}\mathscr{G}$, for $\alpha_{ij}, \beta_{ij} \in \mathscr{G}_{ij}$;

$$\begin{aligned}&V_{ii} \otimes \alpha_{ij} \otimes \beta_{ji} = \alpha_{ij} \otimes \beta_{ji} \otimes V_{ii} \\ &\text{i.e., } (V_{ii} \otimes \alpha_{ij}) \otimes \beta_{ji} = \alpha_{ij} \otimes (\beta_{ji} \otimes V_{ii}) \\ &\text{i.e., } \psi(\alpha_{ij})\, V_{ij} \otimes \beta_{ji} = \alpha_{ij} \otimes \psi(\overline{\beta_{ji}})\, V_{ji} \text{ by assertion (ii)} \\ &\text{i.e., } \psi(\alpha_{ij})\varphi(\beta_{ji})\, V_{ii} = \psi(\overline{\beta_{ji}})\varphi(\bar{\alpha}_{ij})\, V_{ii} \text{ by assertion (i)} \\ &\quad \text{i.e., } \psi(\alpha_{ij})\varphi(\beta_{ji}) = \psi(\overline{\beta_{ji}})\varphi(\bar{\alpha}_{ij}) \\ &\quad \text{i.e., } \psi(\alpha_{ij})C(\overline{\beta_{ji}}) = \psi(\overline{\beta_{ji}})C(\alpha_{ij})\end{aligned}$$

therefore

$$\frac{\psi(\alpha_{ij})}{C(\alpha_{ij})} = \frac{\psi(\overline{\beta_{ji}})}{C(\beta_{ji})}$$

Hence,

$$\frac{\psi(\alpha_{ij})}{C(\alpha_{ij})} = k_{ij}, \text{ a strictly positive constant for all } i, j = 1, 2.$$

where $k_{11} = k_{22} = 1$ and $k_{12} = k_{21}$.

Define,

$$d(\alpha_{ij}) = \sqrt{k_{ij}}\, C(\alpha_{ij}).$$

We will verify that the d defined above is a dimension for $\mathscr{G}$. To begin with, note that since $\overline{V_{12}} = V_{21}$ and $k_{12} = k_{21}$, we have $d(\alpha) = d(\bar{\alpha})$ for all α in $\mathscr{G}$. We need to check that,

$$d(\alpha_{ij})d(\beta_{jk}) = \sum_{\gamma_{ik} \in \mathscr{G}_{ik}} \langle \alpha_{ij} \otimes \beta_{jk}, \gamma_{ik} \rangle d(\gamma_{ik}), \quad \text{for all } i, j, k, = 1, 2.$$

Since d is defined to be the dimension function for the hypergroups the above equation is satisfied for $i = j = k$. The remaining cases are $i = j \neq k$ and $i = k \neq j$ as i, j, k belong to $\{1, 2\}$.

Case (i): $i = j \neq k$.

$$V_{ik} = \sum_{\gamma_{ik} \in \mathscr{G}_{ik}} C(\gamma_{ik})\gamma_{ik}$$

therefore

$$V_{ik} \otimes \beta_{ki} = \sum_{\gamma_{ik} \in \mathscr{G}_{ik}} C(\gamma_{ik})\gamma_{ik} \otimes \beta_{ki}$$

i.e.,

$$\varphi(\beta_{ki})\, V_{ii} = \sum_{\alpha_{ii}\in\mathscr{G}_{ii}} \sum_{\gamma_{ik}\in\mathscr{G}_{ik}} C(\gamma_{ik})\langle \gamma_{ik}\otimes\beta_{ki}, \alpha_{ii}\rangle \alpha_{ii}$$

$$C(\overline{\beta_{ki}}) \sum_{\gamma_{ii}\in\mathscr{G}_{ii}} d(\alpha_{ii})\alpha_{ii} = \sum_{\alpha_{ii}\in\mathscr{G}_{ii}} \sum_{\gamma_{ik}\in\mathscr{G}_{ik}} C(\gamma_{ik})\langle \gamma_{ik}\otimes\beta_{ki}, \alpha_{ii}\rangle \alpha_{ii}$$

therefore

$$d(\alpha_{ii})C(\overline{\beta_{ki}}) = \sum_{\gamma_{ik}\in\mathscr{G}_{ik}} \langle \gamma_{ik}\otimes\beta_{ki}, \alpha_{ii}\rangle C(\gamma_{ik})$$

i.e.,

$$d(\alpha_{ii})\frac{1}{\sqrt{k_{ik}}}d(\overline{\beta_{ki}}) = \sum_{\gamma_{ik}\in\mathscr{G}_{ik}} \langle \alpha_{ii}\otimes\overline{\beta_{ki}}, \gamma_{ik}\rangle \frac{1}{\sqrt{k_{ik}}}d(\gamma_{ik})$$

i.e.,

$$d(\alpha_{ii})d(\overline{\beta_{ki}}) = \sum_{\gamma_{ik}\in\mathscr{G}_{ik}} \langle \alpha_{ii}\otimes\overline{\beta_{ki}}, \gamma_{ik}\rangle d(\gamma_{ik}).$$

Case (ii): $i = k \neq j$.

$$V_{ii} = \sum_{\gamma_{ii}\in\mathscr{G}_{ii}} d(\gamma_{ii})\gamma_{ii}$$

$$V_{ii}\otimes\beta_{ij} = \sum_{\gamma_{ii}\in\mathscr{G}_{ii}} d(\gamma_{ii})\gamma_{ii}\otimes\beta_{ij}$$

i.e.,

$$\psi(\beta_{ij})\, V_{ij} = \sum_{\gamma_{ii}\in\mathscr{G}_{ii}} d(\gamma_{ii})\gamma_{ii}\otimes\beta_{ij} \text{ by the assertion (ii)}$$

i.e.,

$$k_{ij}C(\beta_{ij}) \sum_{\gamma_{ij}\in\mathscr{G}_{ij}} C(\alpha_{ij})\alpha_{ij} = \sum_{\alpha_{ij}\in\mathscr{G}_{ij}} \sum_{\gamma_{ii}\in\mathscr{G}_{ii}} d(\gamma_{ii})\langle \gamma_{ii}\otimes\beta_{ij}, \alpha_{ij}\rangle \alpha_{ij}$$

i.e.,

$$k_{ij}C(\alpha_{ij})C(\beta_{ij}) = \sum_{\gamma_{ii}\in\mathscr{G}_{ii}} \langle \alpha_{ij}\otimes\overline{\beta_{ij}}, \gamma_{ii}\rangle d(\gamma_{ii})$$

So

$$d(\alpha_{ij})d(\overline{\beta_{ij}}) = d(\alpha_{ij})d(\beta_{ij}) = \sum_{\gamma_{ii}\in\mathscr{G}_{ii}} \langle \alpha_{ij}\otimes\overline{\beta_{ij}}, \gamma_{ii}\rangle d(\gamma_{ii}).$$

Uniqueness

Let $\tilde{d}$ be a dimension function for $\mathscr{G}$. By the definition of a dimension function, restricted to the hypergroups $\mathscr{G}_{ii}$, $\tilde{d}$ is a dimension function for $\mathscr{G}_{ii}$ and hence equal to the unique dimension function for $\mathscr{G}_{ii}$ which is equal to d, so that d and $\tilde{d}$ agree on $\mathscr{G}_{ii}$, $i = 1, 2$.

So to prove the uniqueness of the dimension function it is enough to show that d and $\tilde{d}$ agree on $\mathscr{G}_{ij}$, $i, j\in\{1,2\}$ and $i \neq j$.

Let $\tilde{V}$ be the element of $\mathbb{C}\mathscr{G}_{ij}$ whose αth coordinate is $\tilde{d}(\alpha_{ij})$, $i \neq j$. Thus $\tilde{V}$ is a vector with strictly positive coordinates. *Assertion*: $\tilde{V}$ is an eigenvector of $\pi^i_j(e_i)$.

Proof of the Assertion: We have

$$
\begin{aligned}
(\pi_j^i(e_i)\,\tilde{V})(\beta_{ij}) &= \sum_{\gamma_{ij}\in\mathcal{G}_{ij}} (\pi_j^i(e_i))(\beta_{ij},\gamma_{ij})\,\tilde{V}(\gamma_{ij})\\
&= \sum_{\gamma_{ij}\in\mathcal{G}_{ij}} \frac{1}{\sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii})^2} \sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii})\pi_j^i(\alpha_{ii})(\beta_{ij},\gamma_{ij})\tilde{d}(\gamma_{ij})\\
&= \frac{1}{\sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii})^2} \sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii}) \sum_{\gamma_{ij}\in\mathcal{G}_{ij}} \langle \alpha_{ii}\otimes\gamma_{ij},\beta_{ij}\rangle \tilde{d}(\gamma_{ij})\\
&= \frac{1}{\sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii})^2} \sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii}) \sum_{\gamma_{ij}\in\mathcal{G}_{ij}} \langle \overline{\alpha_{ii}}\otimes\beta_{ij},\gamma_{ij}\rangle \tilde{d}(\gamma_{ij})\\
&= \frac{1}{\sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii})^2} \sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii})\tilde{d}(\overline{\alpha_{ii}})\tilde{d}(\beta_{ij}).
\end{aligned}
$$

Since $\tilde{d}(\alpha_{ii}) = d(\alpha_{ii}) = d(\overline{\alpha_{ii}})$ for all i, the right hand side of the above becomes

$$
\begin{aligned}
&= \frac{1}{\sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii})^2} \sum_{\alpha_{ii}\in\mathcal{G}_{ii}} d(\alpha_{ii})^2\tilde{d}(\beta_{ij})\\
&= \tilde{d}(\beta_{ij})\\
&= \tilde{V}(\beta_{ij}).
\end{aligned}
$$

Hence $\tilde{V}$ is the Perron–Frobenius eigenvector of $\pi_j^i(e_i)$ so that $\tilde{V}$ is a positive multiple of V_{ij}.

Thus we get, $\tilde{d}(\alpha_{ij}) = Cd(\alpha_{ij})$ and $\tilde{d}(\alpha_{ji}) = C'd(\alpha_{ij})$ for some positive constants C and C'. By the dimension function equation, we get $CC' = 1$.

Also $\tilde{d}(\alpha) = \tilde{d}(\bar{\alpha}) \Rightarrow C = C'$.

Hence, $C = C' = 1$.

Thus, $\tilde{d} = d$.

Hence there exists at most one dimension function for $\mathcal{G}$. **Q.E.D.**

COROLLARY 13

The dimension function of a finite M_2-graded hypergroup satisfies the following equations:

$$\sum_{\alpha_{11}\in\mathcal{G}_{11}} d(\alpha_{11})^2 = \sum_{\alpha_{12}\in\mathcal{G}_{12}} d(\alpha_{12})^2 = \sum_{\alpha_{22}\in\mathcal{G}_{22}} d(\alpha_{22})^2.$$

Proof. We continue to use the notations of the proof of Theorem 12.

$$V_{11} \otimes V_{12} = \sum_{\alpha_{11} \in \mathscr{G}_{11}} d(\alpha_{11}) \alpha_{11} \otimes V_{12}$$

$$= \sum_{\alpha_{11} \in \mathscr{G}_{11}} d(\alpha_{11})(d(\alpha_{11})\, V_{12}), \quad \text{since } \alpha_{11} \otimes V_{12} = d(\alpha_{11})\, V_{12},$$

$$= \sum_{\alpha_{11} \in \mathscr{G}_{11}} d(\alpha_{11})^2\, V_{12}.$$

Also,

$$V_{11} \otimes V_{12} = V_{11} \otimes \sum_{\alpha_{12} \in \mathscr{G}_{12}} C(\alpha_{12}) \alpha_{12}$$

$$= \sum_{\alpha_{12} \in \mathscr{G}_{12}} C(\alpha_{12})\, V_{11} \otimes \alpha_{12}$$

$$= \sum_{\alpha_{12} \in \mathscr{G}_{12}} C(\alpha_{12}) \psi(\alpha_{12})\, V_{12}$$

$$= \sum_{\alpha_{12} \in \mathscr{G}_{12}} d(\alpha_{12})^2\, V_{12}.$$

Thus,

$$\sum_{\alpha_{11} \in \mathscr{G}_{11}} d(\alpha_{11})^2 = \sum_{\alpha_{12} \in \mathscr{G}_{12}} d(\alpha_{12})^2 \tag{1}$$

Similarly,

$$\sum_{\alpha_{22} \in \mathscr{G}_{22}} d(\alpha_{22})^2 = \sum_{\alpha_{21} \in \mathscr{G}_{21}} d(\alpha_{21})^2. \tag{2}$$

But,

$$\sum_{\alpha_{21} \in \mathscr{G}_{21}} d(\alpha_{21})^2 = \sum_{\alpha_{12} \in \mathscr{G}_{12}} d(\alpha_{12})^2,$$

since $d(\alpha) = d(\bar{\alpha})$ and $\alpha \mapsto \bar{\alpha}$ is a bijection between $\mathscr{G}_{12}$ and $\mathscr{G}_{21}$. Thus from (1) and (2),

$$\sum_{\alpha_{11} \in \mathscr{G}_{11}} d(\alpha_{11})^2 = \sum_{\alpha_{12} \in \mathscr{G}_{12}} d(\alpha_{12})^2 = \sum_{\alpha_{22} \in \mathscr{G}_{22}} d(\alpha_{22})^2.$$

Q.E.D.

Remark 14. In [O1], Ocneanu calls the number $\sum_{\alpha_{ij} \in \mathscr{G}_{ij}} d(\alpha_{ij})^2$ "global index" in the context of fusion algebras.

COROLLARY 15

For all α in $\mathscr{G}$, $d(\alpha) < 2$ implies $d(\alpha) = 2\cos\frac{\pi}{n}$ for some integer $n > 2$.

Proof. We have,

$$R_{\alpha_{ji}} V_{ij} = \varphi(\alpha_{ji})\, V_{ii} \tag{3}$$

and

$$R_{\alpha_{ij}} V_{ii} = \psi(\alpha_{ij})\, V_{ij} \tag{4}$$

From (3),

$$R_{\bar{\alpha}_{ij}} V_{ij} = \varphi(\overline{\alpha_{ij}})\, V_{ii}$$

Thus,

$$\begin{aligned} R_{\bar{\alpha}_{ij}} R_{\alpha_{ij}} V_{ii} &= R_{\alpha_{ij}}(\psi(\alpha_{ij})\, V_{ij}) \\ &= \psi(\alpha_{ij})\,\varphi(\overline{\alpha_{ij}})\, V_{ii} \\ &= k_{ij} C(\alpha_{ij})^2\, V_{ii} \\ &= d(\alpha_{ij})^2\, V_{ii}. \end{aligned}$$

Hence V_{ii} is a Perron–Frobenius eigenvector of the matrix $R_{\bar{\alpha}_{ij}} R_{\bar{\alpha}_{ij}}$ with the eigenvalue $d(\alpha_{ij})^2$, so that $d(\alpha_{ij})$ is the norm of a non-negative integer matrix. Hence by Kronecker's theorem (i.e., for a finite matrix X over Z, either $\|X\| = 2\cos\frac{\pi}{n}$ for some integer $n \geqslant 2$ or $\|X\| \geqslant 2$), $d(\alpha_{ij}) < 2 \Rightarrow d(\alpha_{ij}) = 2\cos\frac{\pi}{n}$, for some $n > 2$ ($n \neq 2$ as $d(\alpha_{ij}) \neq 0$ for any α_{ij}). **Q.E.D.**

Remark 16. (a) The above corollary rules out several graphs arising as 'principal graphs' (cf. [GHJ]) for the inclusion of II_1 factors. Just to cite an example, consider the following graph with the Perron–Frobenius eigenvalues of the matrices corresponding to the vertices marked.

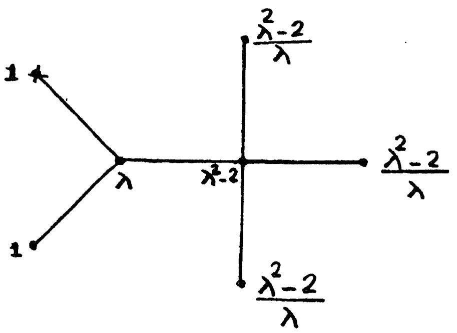
$\frac{\lambda^2-2}{\lambda}$
1
λ
λ^2-2
$\frac{\lambda^2-2}{\lambda}$
1
$\frac{\lambda^2-2}{\lambda}$

Figure 1.

Observe that, $\lambda = (3 + \sqrt{3})^{1/2}$, $\frac{\lambda^2 - 2}{\lambda}$ is between 1 and $\sqrt{2}$ and is not of the form $2\cos(\pi/n)$ for any integer n.

(b) The restriction in the above corollary that we obtained just from the abstract settings of an M_2-graded hypergroup was proved in the case of principal graphs of subfactors in [Po, theorem 3.8] and [I, theorem 6.1].

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On Fatou's lemma and parametric integrals for set-valued functions

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Abstract. In this paper we present new versions of the set-valued Fatou's lemma for sequences of measurable multifunctions and their conditional expectations. Then we use them to study the continuity and measurability properties of parametrized set-valued integrals.

Keywords. Set-valued Fatou's lemma; measurable multifunction; support function; parametrized set-valued integral; weak upper semicontinuity; h^*-upper semicontinuity; lower semicontinuity.

1. Introduction

In the mid-sixties people motivated by applied problems like mathematical economics and control theory, started studying the convergence properties of measurable multifunctions (also known as random sets) and of their integrals. The first significant work in this direction was by Aumann [4], who introduced and studied some of the properties of the integral of multifunctions with values in $\mathbb{R}^n$. Aumann introduced his set-valued integral as an appropriate tool to handle the problem of existence of equilibria for a finite dimensional economy with a continuum of agents. Later Schmeidler [24], motivated by the same problems in mathematical economics and game theory, presented a set-valued version of the well known Fatou's lemma. The work of Schmeidler [24], was extended further by Hildenbrand–Mertens [12] and Artstein [1]. We should also mention the earlier work of Kato [14], who working on a different line of research, namely evolution equations, obtained a result similar to those found in the above mentioned papers on mathematical economics, but for functions with values in a reflexive Banach space whose dual is uniformly convex. Unfortunately this interesting work of Kato was overlooked by people working on mathematical economics. In fact Kato's result is the first set-valued Fatou's lemma in the literature.

In the last decade, the work on evolution inclusions, optimal control and equilibrium theory in economics with infinite dimensional commodity space, necessitated an infinite dimensional version of Fatou's lemma along the line of the above mentioned work of Kato. The first results in this direction were obtained by Hiai [11], Khan–Majumdar [15] and Papageorgiou [19], [20]. Similar results were also obtained later by Balder [6] and Yannelis [28], [29]. Other analogous convergence results for sequences of measurable multifunctions were proved by Pucci–Vitillaro [23] and Papageorgiou [22], using a different mode of set convergence namely weak convergence (i.e. pointwise convergence of support functions). Recently Yannelis [30], examined the implications of Fatou's type results on parametrized set-valued integrals. Such integrals arise naturally in mathematical economics in equilibrium theory. Specifically if (Ω, Σ, μ) is the measure space of agents, X is the commodity space ($\mathbb{R}^n$ or a Banach space), P denotes the price space and $F(\omega, p) \subseteq X$ denotes the demand set for agent $\omega \in \Omega$, when price system $p \in P$ prevails in the market, then the aggregate demand multifunction $\phi(p)$ is given by the set-valued integral of $F(\omega, p)$; i.e. $\phi(p) = \int_\Omega F(\omega, p) \mathrm{d}\mu(\omega)$. The study of the continuity properties of $\phi(\cdot)$, was initiated by Aumann [5], who considered the case $X = \mathbb{R}^n$, while in his very recent work, Yannelis [30] studied the case $X =$ separable Banach space.

The purpose of this paper is to present new versions of the set-valued Fatou's lemma, for both the multifunctions and their set-valued conditional expectations and to improve the results of Yannelis [30] on parametric set-valued integrals.

2. Preliminaries

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this paper we will be using the following notations:

$$P_f(X) = \{A \subseteq X \text{: nonempty, closed}\}$$

and

$$P_{wk(c)}(X) = \{A \subseteq X \text{: nonempty, weakly compact, (convex)}\}.$$

For any $A \in 2^X \setminus \{\emptyset\}$, we set $|A| = \sup\{\|x\| : x \in A\}$ (the "norm" of A), $\sigma(x^*, A) = \sup\{(x^*, a) : a \in A\}$, $x^* \in X^*$ (the "support function of A") and for every $z \in X$ $d(z, A) = \inf\{\|z - a\| : a \in A\}$ (the "distance function from A").

A multifunction $F : \Omega \to 2^X \setminus \{\emptyset\}$ is said to be measurable if for all $U \subseteq X$ nonempty open, $F^-(U) = \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\} \in \Sigma$. If in addition $F(\cdot)$ is $P_f(X)$-valued, then the above definition in equivalent to any of the following two statements:

(i) for every $z \in X$, $\omega \to d(z, F(\omega))$ is measurable,
(ii) there exists $f_n : \Omega \to X$ $n \geqslant 1$ measurable functions s.t. $F(\omega) = \overline{\{f_n(\omega)\}}_{n \geqslant 1}$ for all $\omega \in \Omega$

The above equivalent statements imply the following:

(iii) $\operatorname{Gr} F = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma x B(X)$, with $B(X)$ being the Borel σ-field of X (graph measurability).

If there is a complete, σ-finite measure $\mu(\cdot)$ defined on Σ, then graph measurability is in fact equivalent to measurability for $P_f(X)$-valued multifunctions. For more details on the measurability of multifunctions, we refer to the survey paper of Wagner [26].

In what follows (Ω, Σ, μ) is a finite measure space. Given a measurable multifunction $F : \Omega \to P_f(X)$, by S_F^1 we will denote the set of integrable selectors of $F(\cdot)$; i.e. $S_F^1 = \{f \in L^1(X) : f(\omega) \in F(\omega) \mu\text{-a.e.}\}$. Clearly this set is closed, may be empty and using

Aumann's selection theorem (see Wagner [26], theorem 5.10), it is easy to check that $S_F^1 \neq \emptyset$ if and only if $\omega \to \inf\{\|x\|: x \in F(\omega)\} \in L_+^1$. This is the case if $\omega \to |F(\omega)| \in L_+^1$. Such a multifunction is called "integrably bounded". Using S_F^1, we can define a set-valued integral of $F(\cdot)$, by setting $\int_\Omega F(\omega)\mathrm{d}\mu(\omega) = \left\{ \int_\Omega f(\omega)\mathrm{d}\mu(\omega): f \in S_F^1 \right\}$.

Let Σ_0 be a sub σ-field of Σ and let $F: \Omega \to P_f(X)$ be a Σ-measurable multifunction s.t. $S_F^1 \neq 0$. Following Hiai–Umegaki [10], we define the set-valued conditional expectation of $F(\cdot)$ with respect to Σ_0, to be the Σ_0-measurable multifunction $E^{\Sigma_0}F: \Omega \to P_f(X)$, for which we have $S^1_{E^{\Sigma_0}F} = \mathrm{cl}\{E^{\Sigma_0}f: f \in S_F^1\}$ (the closure taken in the $L^1(\Sigma_0, X)$-norm). If $F(\cdot)$ is integrably bounded (resp. convex valued), then so is $E^{\Sigma_0}F(\cdot)$. Note that in Hiai–Umegaki [10], the definition was given for integrably bounded $F(\cdot)$. However it is clear that it can be extended to the more general class of multifunctions considered here.

Next let $\{A_n\}_{n \geq 1} \subseteq 2^X \setminus \{\emptyset\}$. Following Mosco [18], we define:

$$s\text{-}\underline{\lim}\, A_n = \{x \in X: x = s\text{-}\lim x_n, x_n \in A_n, n \geq 1\} = \{x \in X: \lim \mathrm{d}(x, A_n) = 0\}$$

and

$$w\text{-}\overline{\lim}\, A_n = \{x \in X: x = w\text{-}\lim x_{n_k}, x_{n_k} \in A_{n_k}, n_1 < n_2 < \cdots < n_k < \cdots\}.$$

Here s-denotes the strong topology on X and w- the weak topology on X. It is clear that in general $s\text{-}\underline{\lim}\, A_n \subseteq w\text{-}\overline{\lim}\, A_n$. We say that the A_n's converge to A in the Kuratowski-Mosco sense, denoted by $A_n \xrightarrow{K-M} A$, if and only if $s\text{-}\underline{\lim}\, A_n = w\text{-}\overline{\lim}\, A_n = A$.

Recall that on $P_f(X)$, we can define a generalized metric $h(\cdot,\cdot)$ (known as the Hausdorff metric), by setting

$$h(A, B) = \max\left[\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right], \quad A, B \in P_f(X).$$

Of course $h(\cdot,\cdot)$ is just a pseudometric on $2^X \setminus \{\emptyset\}$. Let $A, B \in 2^X \setminus \{\emptyset\}$. We set

$$h^*(A, B) = \inf\{\varepsilon > 0: A \subseteq B + \varepsilon \mathring{B}_1\},$$

where $\mathring{B}_1 = \{x \in X: \|x\| < 1\}$. Let E be a metric space and let $G: E \to 2^X \setminus \{\emptyset\}$ be a multifunction. We say that $G(\cdot)$ is weakly upper semicontinuous (w-u.s.c.) resp. h^*-u.s.c.) if and only if for every $U \subseteq X$ weakly open, $G^+(U) = \{r \in E: G(r) \subseteq U\}$ is open in E (resp. $r \to h^*(G(r), G(r_0))$ is continuous at r_0). Note that Yannelis [30] uses the name quasi-upper semicontinuous, for h^*-u.s.c. maps. Here we follow the terminology of DeBlasi–Myjak [8]. Also a multifunction $G: E \to 2^X \setminus \{\emptyset\}$ is said to be weakly closed (w-closed) if and only if $\mathrm{Gr}\, G = \{(r, x) \in E \times X: x \in G(r)\}$ is closed in $E \times X_w$ where X_w denotes the Banach space endowed with the weak topology. Recall that for multifunctions with weakly closed values we have, "w-u.s.c. $\Rightarrow$ w-closed" and the two are equivalent if $\overline{G(E)}^w$ is w-compact.

Also we will say that $G: E \to 2^X \setminus \{\emptyset\}$ is lower semicontinuous (l.s.c.) (resp. h^*-l.s.c.) if and only if for every $U \subseteq X$ strongly open, $G^-(U) = \{r \in E: G(r) \cap U \neq \emptyset\}$ is open in E(resp $r \to h^*(G(r_0), G(r))$ is continuous at r_0). An h^*-l.s.c. multifunction is l.s.c. and the two concepts are equivalent if $G(\cdot)$ is compact valued.

Finally let V be a Polish space (i.e. a metrizable, complete separable space). By $M_+^1(V)$ we denote the space of all probability measure on V, endowed with the narrow topology (see Bertsekas–Shreve [7]). Then $M_+^1(V)$ is itself Polish. A map $\mu: E\to M_+^1(V)$ is said to be a transition probability, if for every $A\in B(V)=$ Borel σ-field of V, $r\to\mu(r)(E)$ is Borel-measurable.

3. Set-valued Fatou's lemma

Throughout the rest of the paper (Ω,Σ,μ) will be a finite measure space and X a separable Banach space. Additional hypotheses will be introduced as needed.

We start with a set valued Fatou's type result for the $w\text{-}\overline{\lim}$ of a sequence of measurable multifunctions. Compared with theorem 3.2 of [19], the boundedness hypothesis on the sequence is now weaker, but we have strengthened the hypotheses on the underlying Banach space X and its dual X^*. Also by RNP we will denote the "Radon-Nikodym Property" (see Diestel–Uhl [32]).

Theorem 3.1. *If* $\mu(\cdot)$ *is nonatomic,* X *has the RNP,* X^* *is separable and* $F_n:\Omega\to P_f(X) n\geqslant 1$ *are measurable multifunctions s.t.*

(1) *for every* $A\in\Sigma$, $\overline{\bigcup_{n\geqslant 1}\int_A F_n}^{w}\in P_{wk}(X)$,

(2) $\{|F_n|\}_{n\geqslant 1}$ *is uniformly integrable,*

(3) *for every* $\omega\in\Omega\backslash N$, $\mu(N)=0$, *there exists* $G(\omega)\in P_{wk}(X)$ s.t. $F_n(\omega)\subseteq G(\omega) n\geqslant 1$ then

$$w\text{-}\overline{\lim}\int_\Omega F_n(\omega)d\mu(\omega)\subseteq cl\int_\Omega w\text{-}\overline{\lim}\, F_n(\omega)d\mu(\omega)$$

Proof. Let $x\in w\text{-}\overline{\lim}\int_\Omega F_n(\omega)d\mu(\omega)$. Then by definition we can find $x_k\in\int_\Omega F_{n_k}(\omega)\mathrm{d}\mu(\omega)$ s.t. $x_k\xrightarrow{w}x$ as $k\to\infty$. Let $f_k\in S^1_{F_{n_k}}$ s.t. $x_k=\int_\Omega f_k(\omega)\mathrm{d}\mu(\omega)$. From theorem 1, p. 101 of Diestel–Uhl [32], we know that $\{f_k\}_{k\geqslant 1}$ is relatively sequentially weakly compact. Thus by passing to a subsequence if necessary, we may assume that $f_k\xrightarrow{w}f$ in $L^1(X)$. Then theorem 3.1 of [19] tells us that

$$f(\omega)\in\overline{\mathrm{conv}}\, w\text{-}\overline{\lim}\,\{f_k(\omega)\}_{n\geqslant 1}\subseteq\overline{\mathrm{conv}}\, w\text{-}\overline{\lim}\, F_{n_k}(\omega)\,\mu\text{-a.e.}$$

So

$$x=\int_\Omega f(\omega)\mathrm{d}\mu(\omega)\in\int_\Omega\overline{\mathrm{conv}}\, w\text{-}\overline{\lim}\, F_{n_k}(\omega)\mathrm{d}\mu(\omega).$$

But from proposition 4.2 of Hess [9], we know that $\omega\to\overline{\mathrm{conv}}\, w\text{-}\overline{\lim}\, F_{n_k}(\omega)$ is a $P_{wkc}(X)$-valued, integrably bounded multifunction. So from theorem 3.1 of Kandilakis–Papageorgiou [13], we have

$$\begin{aligned}\int_\Omega\overline{\mathrm{conv}}\, w\text{-}\overline{\lim}\, F_{n_k}(\omega)\mathrm{d}\mu(\omega)&=cl\int_\Omega w\text{-}\overline{\lim}\, F_{n_k}(\omega)\mathrm{d}\mu(\omega)\\&\subseteq cl\int_\Omega w\text{-}\overline{\lim}\, F_n(\omega)\mathrm{d}\mu(\omega)\end{aligned}$$

$$\Rightarrow x \in cl \int_\Omega \text{w-lim}\, F_n(\omega) d\mu(\omega)$$

$$\Rightarrow \text{w-}\overline{\lim} \int_\Omega F_n(\omega) d\mu(\omega) \subseteq cl \int_\Omega \text{w-}\overline{\lim}\, F_n(\omega) d\mu(\omega) \qquad \textbf{Q.E.D.}$$

Remark. If we drop the nonatomicity hypothesis on $\mu(\cdot)$, then the conclusion of the above theorem will be as follows: "$\text{w-}\overline{\lim} \int_\Omega F_n(\omega) d\mu(\omega) \subseteq \int_\Omega \overline{\text{conv}}\, \text{w-}\overline{\lim}\, F_n(\omega) d\mu(\omega)$".

We have an analogous result for the sequence of the set-valued conditional expectations of the multifunctions with respect to a sub σ-field Σ_0 of Σ. Our result extends theorem 2.4 of Hiai [11], who assumed that X is a separable reflexive Banach space.

Theorem 3.2. *If $F_n : \Omega \rightarrow P_{wk}(X)$ are measurable multifunctions s.t. $F_n(\omega) \subseteq G(\omega)\, \mu$-a.e. with $G : \Omega \rightarrow P_{wkc}(X)$ integrably bounded,*

then $\text{w-}\overline{\lim}\, E^{\Sigma_0} F_n(\omega) \subseteq \overline{\text{conv}}\, E^{\Sigma_0} \text{w-}\overline{\lim}\, F_n(\omega)\, \mu$-a.e.

Proof. From theorem 5 of Klei [16], we know that for all $n \geqslant 1$, all $x^* \in X^*$ and all $A \in \Sigma_0$, we have:

$$\int_A \sigma(x^*, E^{\Sigma_0} F_n(\omega)) d\mu(\omega) = \int_A E^{\Sigma_0} \sigma(x^*, F_n(\omega)) d\mu(\omega).$$

$$\Rightarrow \sigma(x^*, E^{\Sigma_0} F_n(\omega)) = E^{\Sigma_0} \sigma(x^*, F_n(\omega)) \text{ for all } \omega \in \Omega \backslash N, \mu(N) = 0, x^* \in D,$$

with $D \subseteq X^*$ being a countable set which is dense in X^* for the Mackey topology $m(X^*, X)$ (such a set exists since by hypothesis X is separable; see Willansky [27], p. 144). But recall that $E^{\Sigma_0} \overline{\text{conv}}\, F_n(\cdot)$ is $P_{wkc}(X)$-valued (see theorem 5 in [16]) and $\overline{\text{conv}}\, E^{\Sigma_0} F_n = E^{\Sigma_0} \overline{\text{conv}}\, F_n$. So $x^* \rightarrow \sigma(x^*, E^{\Sigma_0} \overline{\text{conv}}\, F_n(\omega)) = \sigma(x^*, E^{\Sigma_0} F_n(\omega))$ is m-continuous. Therefore by a simple density argument we deduce that for all $n \geqslant 1$, all $\omega \in \Omega \backslash N$, $\mu(N) = 0$ and all $x^* \in X^*$, we have

$$\sigma(x^*, E^{\Sigma_0} F_n(\omega)) = E^{\Sigma_0} \sigma(x^*, F_n(\omega)).$$

Let $\phi(\omega, x^*) = \overline{\lim}\, \sigma(x^*, E^{\Sigma_0} F_n(\omega)) = \overline{\lim}\, E^{\Sigma_0} \sigma(x^*, F_n(\omega))$ for $(\omega, x^*) \in (\Omega \backslash N) \times X^*$, $\mu(N) = 0$. For $(\omega, x^*) \in N \times X^*$ we set $\phi(\omega, x^*) = 0$. Then clearly $\phi(\cdot, x^*)$ is measurable with $\phi(\omega, \cdot)$ sublinear and $|\phi(\omega, x^*)| \leqslant \sigma(x^*, E^{\Sigma_0} G(\omega))$ for all $\omega \in \Omega \backslash N$, $\mu(N) = 0$ and all $x^* \in X^*$. Thus for μ-almost all $\omega \in \Omega$, $\phi(\omega, \cdot)$ is m-continuous. Similarly set $\psi(\omega, x^*) = \overline{\lim}\, \sigma(x^*, F_n(\omega))$ and deduce that $\psi(\cdot, x^*)$ is measurable, while $\psi(\omega, \cdot)$ is sublinear and $|\psi(\omega, x^*)| \leqslant \sigma(x^*, G(\omega))$ for all $x^* \in X^*$ and almost all $\omega \in \Omega$. Thus for μ-almost all $\omega \in \Omega$, $\psi(\omega, \cdot)$ is m-continuous. Invoking theorem 6.5.6 (d) of Ash [2], we have

$$\overline{\lim}\, \sigma(x^*, E^{\Sigma_0} F_n(\omega)) = \overline{\lim}\, E^{\Sigma_0} \sigma(x^*, F_n(\omega)) \leqslant E^{\Sigma_0} \overline{\lim}\, \sigma(x^*, F_n(\omega))\, \mu\text{-a.e.}$$

$$\Rightarrow \phi(\omega, x^*) \leqslant E^{\Sigma_0} \psi(\omega, x^*)$$

for all $\omega \in \Omega \backslash N_1, \mu(N_1) = 0$ and all $x^* \in D$.

But note that since $\psi(\omega, x^*)$ is measurable in $\omega \in \Omega$, m-continuous and sublinear in x^*, we can find $H : \Omega \rightarrow P_{wkc}(X)$ integrably bounded multifunction s.t.

$$\psi(\omega, x^*) = \sigma(x^*, H(\omega)) \Rightarrow E^{\Sigma_0} \psi(\omega, x^*) = E^{\Sigma_0} \sigma(x^*, H(\omega))$$

$$= \sigma(x^*, E^{\Sigma_0} H(\omega))\, \mu\text{-a.e..}$$

Thus $E^{\Sigma_0}\psi(\omega,\cdot)$ is m-continuous for μ-almost all $\omega\in\Omega$ and so once again via a simple density argument we get

$$\phi(\omega,x^*)\leqslant E^{\Sigma_0}\psi(\omega,x^*)$$

for all $(\omega,x^*)\in(\Omega\backslash N_2)\times X^*, \mu(N_2)=0$. Also from proposition 3.1 of [19], we have:

$$\begin{aligned} E^{\Sigma_0}\psi(\omega,x^*) &= E^{\Sigma_0}\overline{\lim}\,\sigma(x^*,F_n(\omega))\leqslant E^{\Sigma_0}\sigma(x^*,\text{w-}\overline{\lim}\,F_n(\omega)) \\ &= \sigma(x^*,E^{\Sigma_0}\text{w-}\overline{\lim}\,F_n(\omega))\,\mu\text{-a.e.} \end{aligned}$$

the last equality following from proposition 4.3 of Hess [9], which tells us that $\omega\to\text{w-}\overline{\lim}\,F_n(\omega)$ is a $P_{wk}(X)$-valued, integrably bounded multifunction. So we have

$$\phi(\omega,x^*)=\overline{\lim}\,\sigma(x^*,E^{\Sigma_0}F_n(\omega))\leqslant\sigma(x^*,E^{\Sigma_0}\text{w-}\overline{\lim}\,F_n(\omega))\,\mu\text{-a.e.}$$

This then through proposition 4.1 of [19], shows that

$$\text{w-}\overline{\lim}\,E^{\Sigma_0}F_n(\omega)\subseteq\overline{\text{conv}}\,E^{\Sigma_0}\text{w-}\overline{\lim}\,F_n(\omega)=E^{\Sigma_0}\overline{\text{conv}}\,\text{w-}\overline{\lim}\,F_n(\omega)\,\mu\text{-a.e.}.$$

Q.E.D.

We can improve the conclusion of the above theorem with an extra hypothesis on the measure space.

Theorem 3.3. *If $\mu(\cdot)$ has no Σ_0-atoms and $F_n:\Omega\to P_{wk}(X)$ are measurable multifunctions s.t. $F_n(\omega)\subseteq G(\omega)\,\mu$-a.e. with $G:\Omega\to P_{wkc}(X)$ integrably bounded, then w-$\overline{\lim}\,E^{\Sigma_0}F_n(\omega)\subseteq E^{\Sigma_0}\text{w-}\overline{\lim}\,F_n(\omega)\,\mu$-a.e.*

Proof. From proposition 4.3 of Hess [9], we have that $\omega\to\text{w-}\overline{\lim}\,F_n(\omega)$ is $P_{wk}(X)$-valued and integrably bounded. Then theorem 2 of Valadier [25], tells us that $E^{\Sigma_0}\text{w-}\overline{\lim}\,F_n(\cdot)$ is $P_{wkc}(X)$-valued and integrably bounded. So the $\overline{\text{conv}}$ operation is now redundant.

Q.E.D.

We can have a corresponding Fatou's type result for the sequence $\{S^1_{E^{\Sigma_0}F_n}\}_{n\geqslant 1}$. Our result extends theorem 4.2 of [19], where $\Sigma_0=\Sigma$.

Theorem 3.4. *If $\mu(\cdot)$ has no Σ_0-atoms and $F_n:\Omega\to P_{wk}(X) n\geqslant 1$ are measurable multifunctions s.t. $F_n(\omega)\subseteq G(\omega)\,\mu$-a.e. with $G:\Omega\to P_{wkc}(X)$ integrably bounded, then w-$\overline{\lim}\,S^1_{E^{\Sigma_0}F_n}\subseteq S^1_{E^{\Sigma_0}\text{w-}\overline{\lim}\,F_n}$.*

Proof. Let $h\in\text{w-}\overline{\lim}\,S^1_{E^{\Sigma_0}F_n}$. Then we can find $h_n\in S^1_{E^{\Sigma_0}F_n}$ s.t. $h_n\xrightarrow{w}h$ in $L^1(\Sigma_0,X)$. Note that for all $n\geqslant 1$, $h_n(\omega)\in E^{\Sigma_0}G(\omega)\in P_{wkc}(X)\,\mu$-a.e.. So invoking theorem 3.1 of [19], we get

$$\begin{aligned} h(\omega)&\in\overline{\text{conv}}\,\text{w-}\overline{\lim}\,\{h_n(\omega)\}_{n\geqslant 1} \\ &\subseteq\overline{\text{conv}}\,\text{w-}\overline{\lim}\,E^{\Sigma_0}F_n(\omega) \\ &\subseteq E^{\Sigma_0}\text{w-}\overline{\lim}\,F_n(\omega)\,\mu\text{-a.e.} \end{aligned}$$

(see theorem 3.4 and recall that since $\mu(\cdot)$ has no Σ_0-atoms, $E^{\Sigma_0}\text{w-}\overline{\lim}\,F_n(\omega)\in P_{wkc}(X)$ μ-a.e.). Therefore we get $h\in S^1_{E^{\Sigma_0}\text{w-lim}\,F_n}$, establishing the theorem. **Q.E.D.**

We can have a corresponding result for the s-$\underline{\lim}$ of the sequence $\{S^1_{E^{\Sigma_0}F_n}\}_{n\geqslant 1}$. This result extends theorem 4.2 of [19], where $\Sigma_0=\Sigma$ and the boundedness hypothesis on the sequence $\{F_n\}_{n\geqslant 1}$ was a little stronger.

Theorem 3.5. *If* $F_n:\Omega\to P_f(X)$ *are measurable multifunctions s.t.* $d(0,F_n(\omega))\leqslant\phi_n(\omega)$ *μ-a.e. for all* $n\geqslant 1$ *with* $\{\phi_n(\cdot)\}_{n\geqslant 1}$ *uniformly integrable and* $S^1_{\text{s-}\underline{\lim}\,F_n}\neq\emptyset$, *then* $S^1_{E^{\Sigma_0}\text{s-}\underline{\lim}\,F_n}\subseteq$ s-$\underline{\lim}\,S^1_{E^{\Sigma_0}F_n}$.

Proof. Let $h\in S^1_{\text{s-}\underline{\lim}\,F_n}$. Then for every $f\in S^1_{F_n}$, we have

$$\begin{aligned}\int_\Omega \|h(\omega)-f(\omega)\|\,\mathrm{d}\mu(\omega)&=\int_\Omega \|h(\omega)-f(\omega)\|\,\mathrm{d}\mu(\omega)\\&\geqslant\int_\Omega \|E^{\Sigma_0}h(\omega)-E^{\Sigma_0}f(\omega)\|\,\mathrm{d}\mu(\omega)\\&\geqslant\int_\Omega d(E^{\Sigma_0}h(\omega),E^{\Sigma_0}F_n(\omega))\,\mathrm{d}\mu(\omega)\end{aligned}$$

Since $f\in S^1_{F_n}$ was arbitrary, by applying theorem 2.2 of Hiai–Umegaki [10], we get

$$\begin{aligned}&\inf_{f\in S^1_{F_n}}\int_\Omega \|h(\omega)-f(\omega)\|\,\mathrm{d}\mu(\omega)\\&=\int_\Omega \inf_{z\in F_n(\omega)}\|h(\omega)-z\|\,\mathrm{d}\mu(\omega)\\&=\int_\Omega d(h(\omega),F_n(\omega))\,\mathrm{d}\mu(\omega)\geqslant\int_\Omega \mathrm{d}(E^{\Sigma_0}h(\omega),E^{\Sigma_0}F_n(\omega))\,\mathrm{d}\mu(\omega)\\&\qquad\qquad\qquad\qquad\qquad\qquad=d(E^{\Sigma_0}h,S^1_{E^{\Sigma_0}F_n})\end{aligned}$$

But $d(h(\omega),F_n(\omega))\leqslant\|h(\omega)\|+\phi_n(\omega)$ μ-a.e. and $d(h(\omega),F_n(\omega))\to 0$ μ-a.e. So applying the extended dominated convergence theorem (see Ash [2], theorem 7.5.2, p. 295), we get

$$\int_\Omega d(h(\omega),F_n(\omega))\,\mathrm{d}\mu(\omega)\to 0\Rightarrow d(E^{\Sigma_0}h,S^1_{E^{\Sigma_0}F_n})\to 0\text{ as }n\to\infty\Rightarrow E^{\Sigma_0}h\in$$

$$\text{s-}\underline{\lim}\,S^1_{E^{\Sigma_0}F_n}\Rightarrow E^{\Sigma_0}S^1_{s-\lim F_n}\subseteq\text{s-}\underline{\lim}\,S^1_{E^{\Sigma_0}F_n}.$$

Since the latter is closed in $L^1(\Sigma_0,X)$, we conclude that $S^1_{E^{\Sigma_0}\text{s-}\underline{\lim}\,F_n}\subseteq$ s-$\underline{\lim}\,S^1_{E^{\Sigma_0}F_n}$.

Q.E.D.

We can have a pointwise version of the previous result. This theorem was first proved by Hiai [11], but our proof is different.

Theorem 3.6. *If* $F_n:\Omega\to P_f(X)$ *are measurable multifunction s.t.* $d(0,F_n(\omega))\leqslant\phi(\omega)$ *μ-a.e. for all* $n\geqslant 1$ *with* $\phi\in L^1_+$ *and* $S^1_{\text{s-}\underline{\lim}F_n}\neq\emptyset$, *then* $E^{\Sigma_0}\,\text{s-}\underline{\lim}\,F_n(\omega)\subseteq\text{s-}\underline{\lim}\,E^{\Sigma_0}F_n(\omega)$ *μ-a.e..*

Proof. As in the proof of theorem 3.5 above, for every $A\in\Sigma_0$ we can get

$$\int_A d(E^{\Sigma_0}h(\omega),E^{\Sigma_0}F_n(\omega))\mathrm{d}\mu(\omega)\leqslant\int_A d(h(\omega),F_n(\omega))\mathrm{d}\mu(\omega)$$
$$=\int_A E^{\Sigma_0}d(h(\omega),F_n(\omega))\mathrm{d}\mu(\omega)$$

for all $h\in S^1_{s\text{-}\lim F_n}$. So we have for all $n\geqslant 1$,

$$d(E^{\Sigma_0}h(\omega),E^{\Sigma_0}F_n(\omega))\leqslant E^{\Sigma_0}d(h(\omega),F_n(\omega))\,\mu\text{-a.e.}$$
$$\Rightarrow\overline{\lim}\,d(E^{\Sigma_0}h(\omega),E^{\Sigma_0}F_n(\omega))\leqslant\underline{\lim}\,E^{\Sigma_0}d(h(\omega),F_n(\omega))$$
$$\leqslant E^{\Sigma_0}\underline{\lim}\,d(h(\omega),F_n(\omega))=0\,\mu\text{-a.e..}$$

(for the last inequality, see Ash [2], theorem 6.5.6, p. 257). Hence

$$E^{\Sigma_0}h(\omega)\subseteq s\text{-}\underline{\lim}\,E^{\Sigma_0}F_n(\omega)\,\mu\text{-a.e.}$$
$$\Rightarrow E^{\Sigma_0}s\text{-}\underline{\lim}\,F_n(\omega)\subseteq s\text{-}\underline{\lim}\,E^{\Sigma_0}F_n(\omega)\,\mu\text{-a.e..}$$

Q.E.D.

Remark. Note that the uniform boundedness hypothesis by $\phi(\cdot)\in L^1_+$, cannot be weakened to a pointwise boundedness hypothesis by a uniformly integrable sequence $\{\phi_n\}_{n\geqslant 1}\subseteq L^1_+$, as was the case with theorem 3.5. This is because in that case the conditional Fatou's lemma and dominated convergence theorem are not valid. For a counterexample, we refer to Zheng Wei-an [31].

Now we can combine our set-valued Fatou's type results to obtain dominated convergence theorems for set-valued conditional expectations and their sets of integral selectors. Our result extends theorem 4.4 of [19] as well as theorem 3.2 of Yannelis [29], where $\Sigma_0=\Sigma$.

Theorem 3.7. *If $\mu(\cdot)$ has no Σ_0-atoms, $F_n:\Omega\to P_{wk}(X)$ a measurable multifunctions s.t. $F_n(\omega)\subseteq G(\omega)\,\mu$-a.e. with $G:\Omega\to P_{wkc}(X)$ integrably bounded and $F_n(\omega)\xrightarrow{K-M}F(\omega)\,\mu$-a.e., then (1) $E^{\Sigma_0}F_n(\omega)\xrightarrow{K-M}E^{\Sigma_0}F(\omega)\,\mu$-a.e., (2) $S^1_{E^{\Sigma_0}F_n}\xrightarrow{K-M}S^1_{E^{\Sigma_0}F}$.*

Remark. Note that by proposition 4.3 of Hess the limit multifunction $\omega\to F(\omega)$ is immediately $P_{wk}(X)$-valued and integrably bounded.

We also have a dominated convergence theorem for set-valued integrals. Our result gives an alternative version of theorem 3.4 of [19] (different set of hypotheses).

Theorem 3.8. *If X has the RNP, X^* is separable and $F_n:\Omega\to P_f(X)\,n\geqslant 1$ are measurable multifunctions s.t.*

(1) *for all* $A\in\Sigma$, $\overline{\bigcup_{n\geqslant 1}\int_A F(\omega)\mathrm{d}\mu(\omega)}^{w}\in P_{wk}(X)$,

(2) $\{|F_n|\}_{n\geqslant 1}$ *is uniformly integrable,*

(3) *for every* $\omega\in\Omega\backslash N,\mu(N)=0$, *there exists* $G(\omega)\in P_{wkc}(X)$ *s.t.* $F_n(\omega)\subseteq G(\omega)\,n\geqslant 1$,

(4) $F_n(\omega)\xrightarrow{K-M}F(\omega)\,\mu$*-a.e., then* $\int_\Omega F_n(\omega)\mathrm{d}\mu(\omega)\xrightarrow{K-M}\int_\Omega F(\omega)\mathrm{d}\mu(\omega)$.

Proof. Combine theorem 3.1 of this paper with theorem 3.3 of [19]. **Q.E.D.**

4. Parametrized set-valued integrals

In this section we turn our attention to the study of parametrized set-valued integrals. Our work extends the results of Yannelis [30] and produces new ones.

Let E be a complete metric space representing the parameter space. Our first result extends theorem 3.1 of Yannelis [30], who had a pointwise boundedness hypothesis by an L^1_+-function, uniformly with respect to the parameter. In addition our proof is much shorter.

Theorem 4.1. *If* $F:\Omega \times E \to P_f(X)$ *is a multifunction s.t.*

(1) *for every* $r \in E$, $\omega \to F(\omega, r)$ *is measurable,*
(2) *for every* $\omega \in \Omega \setminus N$, $\mu(N) = 0$, $r \to F(\omega, r)$ *is* h^**-u.s.c.,*
(3) *for all* $K \subseteq E$ *compact* $\{|F(\cdot, r)|\}_{r \in K}$ *is uniformly integrable then* $r \to \Gamma(r) = \int_\Omega F(\omega, r) d\mu(\omega)$ *is* h^**-u.s.c..*

Proof. Let $r_n \to r$ in E. Then by definition we have:

$$\int_\Omega h^*(F(\omega, r_n), F(\omega, r)) \mathrm{d}\mu(\omega)$$

$$= \int_\Omega \sup_{x \in F(\omega, r_n)} d(x, F(\omega, r)) \mathrm{d}\mu(\omega)$$

$$= \sup_{h \in S^1_{F(\cdot, r_n)}} \int_\Omega d(h(\omega), F(\omega, r)) \mathrm{d}\mu(\omega)$$

(see theorem 2.2 of Hiai–Umegaki [10]),

$$= \sup_{h \in S^1_{F(\cdot, r_n)}} \int_\Omega \inf_{y \in F(\omega, r)} \|h(\omega) - y\| \mathrm{d}\mu(\omega)$$

$$= \sup_{h \in S^1_{F(\cdot, r_n)}} \inf_{f \in S^1_{F(\cdot, r)}} \int_\Omega \|h(\omega) - f(\omega)\| \mathrm{d}\mu(\omega)$$

$$\geqslant \sup_{h \in S^1_{F(\cdot, r_n)}} \inf_{f \in S^1_{F(\cdot, r)}} \left\| \int_\Omega (h(\omega) - f(\omega)) \mathrm{d}\mu(\omega) \right\|$$

$$= \sup_{w \in \Gamma(r_n)} \inf_{v \in \Gamma(r)} \|w - v\| = h^*(\Gamma(r_n), \Gamma(r)).$$

But note that $h^*(F(\omega, r_n), F(\omega, r)) \leqslant |F(\omega, r_n)| + |F(\omega, r)|$ μ-a.e. $n \geqslant 1$. So from hypothesis (3) with $K = \{r_n, r\}_{n \geqslant 1}$, we have that $\{h^*(F(\cdot, r_n), F(\cdot, r))\}_{n \geqslant 1}$ is uniformly integrable, while from hypothesis (2) we have that $h^*(F(\omega, r_n), F(\omega, r)) \to 0$ μ-a.e. Thus invoking theorem 7.5.2, p. 295 of Ash [2], we get

$$\int_\Omega h^*(F(\omega, r_n), F(\omega, r)) \mathrm{d}\mu(\omega) \to 0 \Rightarrow h^*(\Gamma(r_n), \Gamma(r)) \to 0 \text{ as } n \to \infty,$$

$\Rightarrow \Gamma(\cdot)$ is h^*-u.s.c..

Q.E.D.

The next theorem extends theorem 3.2 of Yannelis [30], since our boundedness hypothesis on the sequence of multifunctions is weaker and in addition our conclusion is stronger.

Theorem 4.2. *If $F:\Omega\times E\to P_{fc}(X)$ is a multifunction s.t.*

(1) *for every $r\in E$, $\omega\to F(\omega,r)$ is measurable,*
(2) *for all $\omega\in\Omega\backslash N, \mu(N)=0$, $r\to F(\omega,r)$ is w-closed,*
(3) *for all $K\subseteq E$ compact, there exists $G_K:\Omega\to P_{wkc}(X)$ integrably bounded s.t. $F(\omega,r)\subseteq G_K(\omega)\mu$-a.e., $r\in K$, then $r\to\Gamma(r)=\int_\Omega F(\omega,r)\mathrm{d}\mu(\omega)$* is *w-u.s.c..*

Proof. To establish the weak upper-semicontinuity of the integral multifunction $\Gamma(\cdot)$, we have to show that given $C\subseteq X$ nonempty, weakly closed, then $\Gamma^-(C)=\{r\in E:\Gamma(r)\cap C\neq\varnothing\}$ is a closed subset of E. So let $\{r_n\}_{n\geqslant 1}\subseteq\Gamma^-(C)$ s.t. $r_n\to r$. Then for every $n\geqslant 1$ we can find $x_n\in\Gamma(r_n)\cap C$. But note that for $K=\{r_n,r\}_{n\geqslant 1}\subseteq E$ compact we have $\{x_n\}_{n\geqslant 1}\subseteq\int_\Omega G_K(\omega)\mathrm{d}\mu(\omega)\in P_{wkc}(X)$ (see the corollary to proposition 3.1 in [21]). So by passing to a subsequence if necessary, we may assume that $x_n\xrightarrow{w}x$. Then $x\in w\text{-}\overline{\lim}\,\Gamma(r_n)=w\text{-}\overline{\lim}\int_\Omega F(\omega,r_n)\mathrm{d}\mu(\omega)\subseteq\int_\Omega\overline{\mathrm{conv}}\,w\text{-}\overline{\lim}\,F(\omega,r_n)\mathrm{d}\mu(\omega)$ (see theorem 3.2 of [13]). But because of hypothesis (2) we have $w\text{-}\overline{\lim}\,F(\omega,r_n)\subseteq F(\omega,r)\mu$-a.e. $\Rightarrow\overline{\mathrm{conv}}\,w\text{-}\overline{\lim}\,F(\omega,r_n)\subseteq F(\omega,r)\mu$-a.e.. So $x\in\int_\Omega F(\omega,r)\mathrm{d}\mu(\omega)=\Gamma(r)$ and clearly $x\in C$. Hence $r\in\Gamma^-(C)\Rightarrow\Gamma^-(C)$ is closed and so $\Gamma(\cdot)$ is w-u.s.c.

Q.E.D.

Remark. An alternative proof of this result is the following:

"From theorem 2.2 of Kandilakis–Papageorgiou [13], we have that for all $x^*\in X^*$ $\sigma(x^*,\Gamma(r))=\int_\Omega\sigma(x^*,F(\omega,r))\mathrm{d}\mu(\omega)$. Invoking Fatou's lemma we can easily check that $r\to\sigma(x^*,\Gamma(r))$ is u.s.c.. Then since $\Gamma(\cdot)$ is $P_{wkc}(X)$-valued the upper-semicontinuity of the support function of $\Gamma(\cdot)$, implies its weak upper-semicontinuity (see theorem 10, p. 128 of Aubin–Ekeland [3])".

Similarly using theorem 3.1 of this paper, we can have the following alternative version of the previous weak upper-semicontinuity result.

Theorem 4.3. *If X has the RNP, X^* is separable and $F:\Omega\times E\to P_{wk}(X)$ is a multifunction s.t. for every $K\subseteq E$ compact, we have*

(1) $\overline{\bigcup_{r\in K}\int_A F(\omega,r)\mathrm{d}\mu(\omega)}^{w}\in P_{wk}(X)$ *for all $A\in\Sigma$,*
(2) *$\{|F(\cdot,r)|\}_{r\in K}$ is uniformly integrable,*
(3) *there exists $G_K:\Omega\to P_{wkc}(X)$ integrably bounded s.t. $F(\omega,r)\subseteq G_k(\omega)$ μ-a.e, $r\in K$,*

$F(\omega,r)\subseteq G_K(\omega)\mu$-a.e., *and* (4) *for all* $\omega \in \Omega \backslash N, \mu(N)=0$, $r \rightarrow F(\omega, r)$ *is w-closed, then* $r \rightarrow \Gamma(r) = \int_{\Omega} F(\omega, r) \mathrm{d}\mu(\omega)$ *is w-u.s.c..*

Proof. The same as that of theorem 4.2, using the version of the set-valued Fatou's lemma is provided by theorem 2.1 of this paper. **Q.E.D.**

We can also allow the integration measure to depend on the parameter (transition measure).

Theorem 4.4. *If* V, E *are Polish spaces,* $\mu: E \rightarrow M_{+}^{1}(V)$ *is a continuous transition probability and* $F: V \times E \rightarrow P_{wkc}(X)$ *is a w-u.s.c. multifunction s.t.* $|F(v, r)| \leqslant M$ *for all* $(v, r) \in V \times E$, *then* $r \rightarrow \Gamma(r) = \int_{\Omega} F(v, r) \mathrm{d}\mu(r)(v)$ *is w-u.s.c..*

Proof. For every $x^* \in X^*$, we have $\sigma(x^*, \Gamma(r)) = \int_{V} \sigma(x^*, F(v, r)) \mathrm{d}\mu(r)(v)$. Since $(v, r) \rightarrow F(v, r)$ is w-u.s.c. $\Rightarrow (v, r) \rightarrow \sigma(x^*, F(v, r))$ is u.s.c. (see proposition 2, p. 122 of Aubin–Ekeland [3]). Then lemma 7.14, p. 147 of Bertsekas–Shreve [7], gives us functions $u_n(x^*): V \times E \rightarrow \mathbb{R}$ continuous s.t. $u_n(x^*)(v, r) \downarrow \sigma(x^*, F(v, r))$ for all $(v, r) \in V \times E$. From Feller's theorem (see for example Bertsekas–Shreve [7], proposition 7.30, p. 145), we have

$$r \rightarrow \int_{V} u_n(x^*)(v, r) \mu(r)(dv) = \eta_n(x^*)(r) n \geqslant 1$$

is continuous. Also from the monotone convergence theorem, we have

$$\eta_n(x^*)(r) \downarrow \sigma(x^*, \Gamma(r))$$
$$\Rightarrow r \rightarrow \sigma(x^*, \Gamma(r)) \text{ is u.s.c..}$$

Since $\Gamma(\cdot)$ is $P_{wkc}(X)$-valued, we conclude that $\Gamma(\cdot)$ is w-u.s.c..

Q.E.D.

Also we can have a lower semicontinuity (resp. h^*-lower semicontinuity) result that extends theorem 3.3 of Yannelis [30].

Theorem 4.5. *If* $F: \Omega \times E \rightarrow 2^X \backslash \{\varnothing\}$ *is multifunction s.t.*

(1) *for every* $r \in E$, $\omega \rightarrow F(\omega, r)$ *is graph measurable,*
(2) *for every* $\omega \in \Omega \backslash N, \mu(N)=0$, $r \rightarrow F(\omega, r)$ *is l.s.c.* (resp. h^*-*l.s.c.*)
(3) *for every* $K \subseteq E$ *compact* $\{|F(\cdot, r)|\}_{r \in K}$ *is uniformly integrable, then* $r \rightarrow \Gamma(r) = \int_{\Omega} F(\omega, r) \mathrm{d}\mu(\omega)$ *is l.s.c. (resp.* h^*-*(l.s.c.).*

Proof. The lower semicontinuity part of the theorem follows from theorem 3.3. of [19], while the h^*-lower semicontinuity follows from the inequality $h^*(\Gamma(r), \Gamma(r_n)) \leqslant$

$\int_\Omega h^*(F(\omega,r),F(\omega,r_n))d\mu(\omega)$, where $r_n \to r$ in E (see the proof of theorem 4.1).

Q.E.D.

Our last results deal with the measurability properties of the parametrized set-valued integral. We start with a "finite dimensional" result.

Theorem 4.6. *If $\mu(\cdot)$ is nonatomic, $(E,\mathscr{S})$ is a measurable space and $F:\Omega \times E \to P_f(\mathbb{R}^m)$ is a measurable multifunction s.t. for every $r \in E$, $F(\cdot,r)$ is integrably bounded then $r \to \Gamma(r) = \int_\Omega F(\omega,r)d\mu(\omega)$ is a measurable multifunction with compact, convex values in $\mathbb{R}^m$.*

Proof. Let $f_n:\Omega \times E \to \mathbb{R}^m n \geq 1$, be measurable maps s.t. $F(\omega,r) = \overline{\{f_n(\omega,r)\}}_{n\geq 1}$. Using theorem 18.3.2, p. 206 of Klein–Thompson [17], we have

$$cl\int_\Omega \{f_n(\omega,r)\}_{n\geq 1}d\mu(\omega) \subseteq \int_\Omega \overline{\{f_n(\omega,r)\}}_{n\geq 1}d\mu(\omega) = \int_\Omega F(\omega,r)d\mu(\omega)$$

and the latter is a compact, convex subset of $\mathbb{R}^m$. On the other hand by corollary 18.1.10, p. 203 of Klein–Thompson [17], we have:

$$\int_\Omega \overline{\text{conv}}\{f_n(\omega,r)\}_{n\geq 1}d\mu(\omega) = cl\int_\Omega \{f_n(\omega,r)\}_{n\geq 1}d\mu(\omega).$$

Hence we deduce that

$$\int_\Omega F(\omega,r)d\mu(\omega) = cl\int_\Omega \{f_n(\omega,r)_{n\geq 1}d\mu(\omega).$$

So for every

$$x \in \mathbb{R}^m d(x,\Gamma(r)) = \inf_{n\geq 1}\left\| x - \int_\Omega f_n(\omega,r)d\mu(\omega)\right\| = \inf_{n\geq 1}\|x - u_n(r)\|,$$

where $u_n(r) = \int_\Omega f_n(\omega,r)d\mu(\omega)$. But $r \to u_n(r) n \geq 1$ is $\mathscr{S}$-measurable. So $r \to d(x,\Gamma(r))$ is measurable $\Rightarrow \Gamma(\cdot)$ is measurable with nonempty, compact, convex values $\mathbb{R}^m$.

Q.E.D.

We also have an infinite dimensional version of this result. Recall that a σ-field $\mathscr{S}$ is said to be complete, if $\mathscr{S} = \hat{\mathscr{S}}$, where $\hat{\mathscr{S}}$ is the universal σ-field generated by $\mathscr{S}$. Note that if we are given a finite measure $\lambda(\cdot)$ with respect to which $\mathscr{S}$ is complete then $\mathscr{S} = \hat{\mathscr{S}}$ (see Wagner [26]).

Theorem 4.7. *If $(E,\mathscr{S})$ is a measurable space with $\mathscr{S}$ complete, $F:\Omega \times E \to P_{wkc}(X)$ is measurable and for all $r \in E$, $F(\cdot,r)$ is integrably bounded, then $r \to \Gamma(r) = \int_\Omega F(\omega,r)d\mu(\omega)$ is a $P_{wkc}(X)$-valued measurable multifunction.*

Proof. Recall that for every $x^* \in X^*$, we have

$$\sigma(x^*, \Gamma(r)) = \int_\Omega \sigma(x^*, F(\omega, r)) \mathrm{d}\mu(\omega)$$

$$\Rightarrow r \to \sigma(x^*, \Gamma(r)) \text{ is measurable.}$$

Since $\mathscr{S} = \hat{\mathscr{S}}$ and $\Gamma(\cdot)$ is $P_{wkc}(X)$-valued (see [21], section 3), we conclude that $\Gamma(\cdot)$ is measurable (see Wagner [26]). **Q.E.D.**

We can drop the completeness hypothesis on $\mathscr{S}$ and instead assume that $\mu(\cdot)$ is nonatomic.

Theorem 4.8. *If μ is nonatomic, $(E, \mathscr{S})$ is a measurable space, $F: \Omega \times E \to P_{wkc}(X)$ is a measurable multifunction s.t. for all $r \in E$, $F(\cdot, r)$ is integrably bounded, then $r \to \Gamma(r) = \int_\Omega F(\omega, r) \mathrm{d}\mu(\omega)$ is a $P_{wkc}(X)$-valued measurable multifunction.*

Proof. From [21] we know that $\int_\Omega F(\omega, r) \mathrm{d}\mu(\omega) \in P_{wkc}(X)$. Also let $f_n: \Omega \times E \to X \; n \geq 1$ be measurable maps s.t. $F(\omega, r) = \overline{\{f_n(\omega, r)\}_{n \geq 1}}$. Using theorem 3.2 of Kandilakis–Papageorgiou [13], we get

$$\int_\Omega F(\omega, r) \mathrm{d}\mu(\omega) = cl \int_\Omega \{f_n(\omega, r)\}_{n \geq 1} \mathrm{d}\mu(\omega)$$

$$\Rightarrow x \to d(x, \Gamma(r)) = \inf_{n \geq 1} \|x - u_n(r)\| \text{ is measurable,}$$

where $u_n(r) = \int_\Omega f_n(\omega, r) \mathrm{d}\mu(\omega)$, $n \geq 1$. So indeed $\Gamma(\cdot)$ measurable. **Q.E.D.**

To illustrate our results we present two simple examples.

First let X be a separable reflexive Banach space (the state space), E a complete metric space (the parameter space) and Y a separable Banach space (the control space). We consider the following parametric family of control systems:

$$\left\{ \begin{array}{l} \dot{x}(t) = A(r)x(t) + B(t)u(t) \\ x(0) = x_0(r) \\ u(t) \in U(t, r) \, a.e. \end{array} \right\}. \qquad (*)_r$$

Here $A(r)(\cdot)$ is the infinitesimal generator of a C_0-semigroup $S(r)(\cdot)$ satisfying $\|S(r)(t)\| \leq Me^{wt}$. Also we will assume that if $r_n \to r$ in E, then $(\lambda I - A(r_n)^*)^{-1} x^* \to (\lambda I - A(r)^*)^{-1} x^*$ for all $x^* \in X^*$. Recall that the adjoint $A(r)^*$ of $A(r)$ also generates a C_0-semigroup, which is fact is equal to $S(r)^*(t)$, the adjoint of $S(r)(t)$. Let $U: T \times E \to P_{wkc}(Y)$ be a multifunction which is measurable in t, w-u.s.c. in r and for all $r \in K \subseteq E$ compact $|U(s, r)| \leq a_K(s)$ *a.e.* with $a_K(\cdot) \in L^1_+$. Finally assume that $B(\cdot) \in L^\infty(T, \mathscr{L}(Y, X))$. By a trajectory of $(*)$, we mean a mild solution of the evolution equation generated by an admissible control $u(\cdot)$. Let $b > 0$ and let $R_b(r)$ be the reachable set at time b of $(*)_r$; i.e. $R_b(r) = \{z \in X : z = x(b),\ x(\cdot) \text{ is a trajectory of } (*)_r\}$. Our claim is that $r \to R_b(r)$ is w-u.s.c. from E into X. Indeed note that $R_t(r) = S(r)(b)x_0(r) + \int_0^b S(r)(b-s)B(s)U(s,r)\mathrm{d}s$. Let $G_b(s,r) = S(r)(b-s)B(s)U(s,r)$. Clearly $s \to G_b(s,r)$ is measurable. We will also show that $r \to G_b(s,r)$ is w-closed. To this end let $(r_n, y_n) \in \operatorname{Gr} G_b(s,\cdot)$ $n \geqslant 1$ and assume that $(r_n, y_n) \to (r, y)$ in $E \times X_w$. Then by definition

$$y_n = S(r_n)(b-s)B(s)u_n,\ u_n \in U(s, r_n).$$

Observe that because of our hypothesis on $U(s,r)$, $\overline{\bigcup_{n \geqslant 1} U(s,r_n)}^{w} \in P_{wk}(Y)$. So by passing to a subsequence if necessary, we may assume that $u_n \xrightarrow{w} u$. Hence $B(s)u_n \xrightarrow{w} B(s)u$ in X. Furthermore from Pazy [33], p. 85, we know that $S(r_n)(b-s)^* x^* \xrightarrow{s} S(r)(b-s)^* x^*$ in X^* for all $x^* \in X^*$. So finally $y_n \xrightarrow{w} S(r)(b-s)B(s)u = y$. Since $u \in w\text{-}\overline{\lim}\, U(s,r_n) \subseteq U(s,r)$, we conclude that $(r, y) \in \operatorname{Gr} G_b(s,\cdot)$. Thus apply theorem 4.2, to get the desired conclusion.

As a second example, let (Ω, Σ, μ) be nonatomic probability space modelling the space of agents (see Aumann [5]). Also, let X be a Banach space with the RNP (Radon–Nikodym property) whose dual is separable (so that X is separable too). Let $F : \Omega \times X^* \to P_{wkc}(X)$ be multifunction, representing the demand for goods of agent $\omega \in \Omega$, when the price system $p \in X^*$ prevails in the market. Assume that $\omega \to F(\omega, p)$ is measurable, $p \to F(\omega,p)$ is closed from X^* into X_w and for every $K \subseteq X^*$ compact $F(\omega, p) \subseteq G_K(\omega)$ with $G_K(\omega) \in P_{wk}(X)$ for all $\omega \in \Omega$. Recall that the integral of a $P_{wkc}(X)$-valued measurable multifunction, is w-compact (see Papageorgiou [21]). So we can apply theorem 3.1 and conclude that the aggregate demand multifunction $p \to \Phi(p) = \int_\Omega F(\omega, p)\mathrm{d}\mu(\omega)$ is u.s.c. from X^* into X_w.

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Banerjee *et al*'s characterization theorem in thermohaline convection and its magnetorotatory extensions

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Abstract. The paper mathematically establishes that magnetorotatory thermohaline convection of the Veronis [7] type cannot manifest itself as oscillatory motions of growing amplitude in an initially bottom heavy configuration if the thermohaline Rayleigh number R_s, the Lewis number τ, the thermal Prandtl number σ, the magnetic Prandtl number σ_1, the Chandrasekhar number Q and the Taylor number T satisfy the inequality

$$R_s \leqslant 4\pi^4\left[1+\frac{\tau}{\sigma\pi^2}\left\{\pi^2-\left(Q\sigma_1+\frac{T}{\pi^2}\right)\right\}\right],$$

when both the boundary surfaces are rigid thus achieving magnetorotatory extension of an important characterization theorem of Banerjee *et al* (1992) on the corresponding hydrodynamic problem. A similar characterization theorem is mathematically established in the context of the magnetorotatory thermohaline convection of the Stern (1960) type.

Keywords. Thermohaline convections; thermohaline Rayleigh number; Veronis type; Stern type

1. Introduction

The recently established characterization theorem of [2], disproving the existence of neutral or unstable oscillatory motions of growing amplitude in a bottom heavy thermohaline convection of the Veronis type whenever the thermohaline Rayleigh number is less than a critical value and a similar theorem for a thermohaline convection of the Stern type [3] has brought a fresh outlook to the subject matter of double diffusive convection and paved the way for further theoretical and experimental investigations in this field of enquiry. The essence of Banerjee *et al*'s characterization theorem is that it provides a classification of the neutral and unstable thermohaline convection of the Veronis and Stern types into two classes namely the bottom heavy class and the top heavy class and distinguishes them by means of characterization theorems which disallow the existence of oscillatory motions in the former class. The extension of these two important theorems in astrophysics and terrestrial physics, wherein the liquid concerned has the property of electrical conduction and magnetic fields and rotations are prevalent, is very much to be sought of in the present context. This paper mathematically analyses the magnetorotatory thermohaline convection of the Veronis and the Stern types wherein a uniform rotation about an axis which is parallel to gravity is superimposed on the corresponding magnetohydrodynamic problem [4] may be regarded as a first step of this scheme of extended investigations. Here it is mathematically shown that magnetorotatory thermohaline convection of Veronis type cannot manifest itself as oscillatory motions of growing amplitude in an initially bottom heavy configuration if the thermohaline Rayleigh number R_s, the Lewis number τ, the thermal Prandtl number σ, the magnetic Prandtl number σ_1, the Chandrasekhar number Q and the Taylor number T satisfy the inequality

$$R_s \leqslant 4\pi^4\left[1+\frac{\tau}{\sigma\pi^2}\left\{\pi^2-\left(Q\sigma_1+\frac{T}{\pi^2}\right)\right\}\right],$$

when both the boundary surfaces are rigid thus achieving magnetorotatory extension of an important theorem of Banerjee *et al* [2] on the corresponding hydrodynamic problem. A similar characterization theorem is mathematically established in the context of the magnetorotatory thermohaline convection of the Stern [6] type.

2. Mathematical formulation and analysis

The relevant governing equations and boundary conditions of magnetorotatory thermohaline convection of the Veronis type are given in [1],

$$(D^2-a^2)\left(D^2-a^2-\frac{p}{\sigma}\right)w = Ra^2\theta - R_s a^2\phi + TD\zeta - QD(D^2-a^2)h_z, \tag{1}$$

$$(D^2-a^2-p)\theta = -w, \tag{2}$$

$$\left(D^2-a^2-\frac{p}{\tau}\right)\phi = -\frac{w}{\tau}, \tag{3}$$

$$\left(D^2-a^2-\frac{p\sigma_1}{\sigma}\right)h_z = -Dw, \tag{4}$$

$$\left(D^2-a^2-\frac{p}{\sigma}\right)\zeta = -QD\xi - Dw, \tag{5}$$

$$\left(D^2-a^2-\frac{p\sigma_1}{\sigma}\right)\xi = -D\zeta, \tag{6}$$

and

$$\left.\begin{aligned} w&=0\\ \theta&=0\\ \phi&=0\\ Dw&=0\\ \zeta&=0\\ D\xi&=0\\ h_z&=0 \end{aligned}\right\} \text{at } z=0 \text{ and } z=1, \tag{7}$$

where z is a real independent variable such that $0 \leqslant z \leqslant 1$ and stands for vertical coordinate, d/dz denotes the derivative with respect to z. The other terms $a^2>0$, $\sigma>0$, $\tau>0$, $R>0$, $T>0$, $Q>0$, and $R_s>0$ are constants. $p=p_r+ip_i$ is a complex constant in general and stands for the complex growth rate such that p_r and p_i are real constants and as a consequence the dependent variables $w(z) = w_r(z) + iw_i(z)$, $\theta(z) = \theta_r(z) + i\theta_i(z)$, $\phi(z) = \phi_r(z) + i\phi_i(z)$, $\zeta(z) = \zeta_r(z) + i\zeta_i(z)$, $\xi(z) = \xi_r(z) + i\xi_i(z)$ and $h_z(z) = h_{zr}(z) + ih_{zi}(z)$ which respectively stand for the vertical velocity, temperature, concentration, vertical vorticity, vertical current density and vertical magnetic field and are complex valued functions of the real variable z such that $w_r(z)$ $w_i(z)$, $\theta_r(z)$, $\theta_i(z)$, $\phi_r(z)$, $\phi_i(z)$, $\zeta_r(z)$, $\zeta_i(z)$, $\xi_r(z)$, $\xi_i(z)$, $h_{zr}(z)$ and $h_{zi}(z)$ are real valued functions of real variable z. Equations (1)–(7) describe the eigenvalue problem for p for given values of the other parameters and govern the magnetorotatory thermohaline convection of the Veronis type when both the boundaries are rigid. We prove the following theorems:

Theorem 1. *If* $R_s > 0$, $R > 0$, $Q > 0$, $T > 0$, $p_r \geqslant 0$, $p_i \neq 0$ $\dfrac{Q\sigma_1}{\pi^2} + \dfrac{T}{\pi^4} \leqslant \dfrac{\tau}{\sigma}(\sigma + \tau)$ and $R_s \leqslant 4\pi^4\left[1 + \dfrac{\tau}{\sigma\pi^2}\left\{\pi^2 - \left(Q\sigma_1 + \dfrac{T}{\pi^2}\right)\right\}\right]$ *then a necessary condition for the existence of a non-trivial solution* $(w, \theta, \phi, h_z, \zeta, \xi, p)$ *of equations* (1)–(5) *and* (6) *together with boundary conditions* (7) *is that*

$$R_s < R.$$

Proof. Multiplying equation (1) by w^* (the complex conjugate of w) throughout and integrating the resulting equation over the vertical range of z, we get

$$\int_0^1 w^*(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right)w\mathrm{d}z = Ra^2\int_0^1 w^*\theta\mathrm{d}z - R_s a^2\int_0^1 w^*\phi\mathrm{d}z$$

$$+ T\int_0^1 w^* D\zeta\mathrm{d}z - Q\int_0^1 w^* D(D^2 - a^2)h_z\mathrm{d}z, \tag{8}$$

using equations (2)–(6) and boundary conditions on w and ζ namely $w(0) = 0 = w(1)$ and $\zeta(0) = 0 = \zeta(1)$, we can write

$$Ra^2\int_0^1 w^*\theta\mathrm{d}z = -Ra^2\int_0^1 \theta(D^2 - a^2 - p^*)\theta^*\mathrm{d}z, \tag{9}$$

$$-R_s a^2\int_0^1 w^*\phi\mathrm{d}z = R_s a^2\tau\int_0^1 \phi\left(D^2 - a^2 - \frac{p^*}{\tau}\right)\phi^*\mathrm{d}z, \tag{10}$$

$$-\int_0^1 w^* D(D^2 - a^2)h_z\mathrm{d}z = \int_0^1 Dw^*(D^2 - a^2)h_z\mathrm{d}z = -\int_0^1 h_z(D^2 - a^2)\left(D^2 - a^2 - \frac{p^*\sigma_1}{\sigma}\right)h_z^*\mathrm{d}z, \tag{11}$$

and

$$\int_0^1 w^* D\zeta\mathrm{d}z = -\int_0^1 Dw^*\zeta\mathrm{d}z = \int_0^1 \zeta\left(D^2 - a^2 - \frac{p^*}{\sigma}\right)\zeta^*\mathrm{d}z + Q\int_0^1 \zeta D\xi^*\mathrm{d}z$$

$$= \int_0^1 \zeta\left(D^2 - a^2 - \frac{p^*}{\sigma}\right)\zeta^*\mathrm{d}z - Q\int_0^1 \mathrm{D}\zeta\xi^*\mathrm{d}z$$

$$= \int_0^1 \zeta\left(D^2 - a^2 - \frac{p^*}{\sigma}\right)\zeta^*\mathrm{d}z + Q\int_0^1 \xi\left(D^2 - a^2 - \frac{p\sigma_1}{\sigma}\right)\xi^*\mathrm{d}z, \tag{12}$$

so that using equations (9)–(12), it follows from equation (8) that

$$\int_0^1 w^*(D^2-a^2)\left(D^2-a^2-\frac{p}{\sigma}\right)w\mathrm{d}z = -Ra^2\int_0^1 \theta(D^2-a^2-p^*)\theta^*\mathrm{d}z$$

$$+R_s a^2\tau\int_0^1 \phi\left(D^2-a^2-\frac{p^*}{\tau}\right)\phi^*\mathrm{d}z - Q\int_0^1 h_z(D^2-a^2)$$

$$\left(D^2-a^2-\frac{p^*\sigma_1}{\sigma}\right)h_z^*\mathrm{d}z$$

$$+T\int_0^1 \zeta\left(D^2-a^2-\frac{p^*}{\sigma}\right)\delta^*\mathrm{d}z + QT\int_0^1 \xi\left(D^2-a^2-\frac{p\sigma_1}{\sigma}\right)\xi^*\mathrm{d}z. \qquad (13)$$

Integrating various terms of equation (13) by parts for an appropriate number of times and making use of boundary conditions, we have

$$\int_0^1 [|D^2w|^2+2a^2|Dw|^2+a^4|w|^2]\mathrm{d}z + \frac{p}{\sigma}\int_0^1 [|Dw|^2+a^2|w|^2]\mathrm{d}z$$

$$= Ra^2\int_0^1 [|D\theta|^2+a^2|\theta|^2+p^*|\theta|^2]\mathrm{d}z$$

$$-R_s a^2\tau\int_0^1 \left[|D\phi|^2+a^2|\phi|^2+\frac{p^*}{\tau}|\phi|^2\right]\mathrm{d}z$$

$$-Q\int_0^1 |(D^2-a^2)h_z|^2\mathrm{d}z - \frac{Qp^*\sigma_1}{\sigma}\int_0^1 [|Dh_z|^2+a^2|h_z|^2]\mathrm{d}z$$

$$-T\int_0^1 \left[|D\zeta|^2+a^2|\zeta|^2+\frac{p^*}{\sigma}|\zeta|^2\right]\mathrm{d}z$$

$$-QT\int_0^1 \left[|D\xi|^2+a^2|\xi|^2+\frac{p\sigma_1}{\sigma}|\xi|^2\right]dz \qquad (14)$$

and equating the real and imaginary parts of both sides of equation (15) and cancelling $pi(\neq 0)$ throughout from the imaginary part, we get

$$\int_0^1 [|D^2w|^2+2a^2|Dw|^2+a^4|w|^2]\mathrm{d}z + \frac{pr}{\sigma}\int_0^1 [|Dw|^2+a^2|w|^2]\mathrm{d}z$$

$$= Ra^2\int_0^1 [|D\theta|^2+a^2|\theta|^2+pr|\theta|^2]\mathrm{d}z$$

$$-R_s a^2\tau\int_0^1 \left[|D\phi|^2+a^2|\phi|^2+\frac{pr}{\tau}|\phi|^2\right]\mathrm{d}z$$

$$\div Q\int_0^1 |(D^2-a^2)h_z|^2\mathrm{d}z - \frac{Qpr\sigma_1}{\sigma}\int_0^1 [|Dh_z|^2+a^2|h_z|^2]\mathrm{d}z$$

$$-QT\int_0^1 \left[|D\xi|^2+a^2|\xi|^2+\frac{pr\sigma_1}{\sigma}|\xi|^2\right]\mathrm{d}z -$$

$$T\int_0^1\left[|D\zeta|^2+a^2|\zeta|^2+\frac{pr}{\sigma}|\zeta|^2\right]\mathrm{d}z, \tag{15}$$

and

$$\frac{1}{\sigma}\int_0^1[|Dw|^2+a^2|w|^2]\mathrm{d}z=-Ra^2\int_0^1|\theta|^2\mathrm{d}z+R_sa^2\int_0^1|\phi|^2\mathrm{d}z$$
$$+\frac{Q\sigma_1}{\sigma}\int_0^1[|Dh_z|^2+a^2|h_z|^2]\mathrm{d}z+\frac{T}{\sigma}\int_0^1|\zeta|^2\mathrm{d}z-\frac{QT\sigma_1}{\sigma}\int_0^1|\xi|^2\mathrm{d}z, \tag{16}$$

of which equation (16) can be rearranged in the form

$$\int_0^1[|D^2w|^2+2a^2|Dw|^2+a^4|w|^2]\mathrm{d}z+\frac{pr}{\sigma}\int_0^1[|Dw|^2+a^2|w|^2]\mathrm{d}z$$
$$+Q\int_0^1|(D^2-a^2)h_z|^2\mathrm{d}z+QT\int_0^1[|D\xi|^2+a^2|\xi|^2]\mathrm{d}z$$
$$=Ra^2\int_0^1[|D\theta|^2+a^2|\theta|^2]dz-R_sa^2\tau\int_0^1[|D\phi|^2+a^2|\phi|^2]\mathrm{d}z$$
$$-T\int_0^1[|D\zeta|^2+a^2|\zeta|^2]+pr\left[Ra^2\int_0^1|\theta|^2\mathrm{d}z-R_sa^2\int_0^1|\phi|^2\mathrm{d}z\right.$$
$$\left.-\frac{Q\sigma_1}{\sigma}\int_0^1[|Dh_z|^2+a^2|h_z|^2]\mathrm{d}z-\frac{T}{\sigma}\int_0^1|\zeta|^2\mathrm{d}z+\frac{QT\sigma_1}{\sigma}\int_0^1|\xi|^2\mathrm{d}z.\right. \tag{17}$$

We first note that w, θ, ϕ, ζ and h_z satisfy $w(0)=0=w(1)$, $\theta(0)=0=\theta(1)$, $\phi(0)=0=\phi(1)$, $\zeta(0)=0=\zeta(1)$ and $h_z(0)=0=h_z(1)$ in addition to satisfying the governing equations and hence we have from the Rayleigh–Ritz inequality [5]

$$\int_0^1|Dw|^2\mathrm{d}z\geqslant\pi^2\int_0^1|w|^2\mathrm{d}z, \tag{18}$$

$$\int_0^1|D\theta|^2\mathrm{d}z\geqslant\pi^2\int_0^1|\theta|^2\mathrm{d}z, \tag{19}$$

$$\int_0^1|D\phi|^2\mathrm{d}z\geqslant\pi^2\int_0^1|\phi|^2\mathrm{d}z, \tag{20}$$

$$\int_0^1|D\zeta|^2\mathrm{d}z\geqslant\pi^2\int_0^1|\zeta|^2\mathrm{d}z,$$

and

$$\int_0^1|Dh_z|^2\mathrm{d}z\geqslant\pi^2\int_0^1|h_z|^2\mathrm{d}z, \tag{22}$$

Further, since $w(0)=0=w(1)$, it follows that

$$\int_0^1|Dw|^2\mathrm{d}z=\text{Real part of}\left[-\int_0^1 w^*D^2w\mathrm{d}z\right],$$
$$\leqslant\left|-\int_0^1 w^*D^2w\mathrm{d}z\right|,$$

$$\leqslant \left| \int_0^1 w^* D^2 w \mathrm{d}z \right|,$$

$$\leqslant \int_0^1 |w^*||D^2 w| \mathrm{d}z,$$

$$\leqslant \int_0^1 |w||D^2 w| \mathrm{d}z,$$

$$\leqslant \left[\int_0^1 |w|^2 \mathrm{d}z \right]^{1/2} \left[\int_0^1 |D^2 w|^2 \mathrm{d}z \right]^{1/2}$$

(utilizing Schwartz-inequality),

$$\leqslant \frac{1}{\pi} \left[\int_0^1 |Dw|^2 \mathrm{d}z \right]^{1/2} \left[\int_0^1 |D^2 w|^2 \mathrm{d}z \right]^{1/2}.$$

(utilizing inequality (18)),

so that we have

$$\left[\int_0^1 |Dw|^2 \mathrm{d}z \right]^{1/2} \leqslant \frac{1}{\pi} \left[\int_0^1 |D^2 w|^2 \mathrm{d}z \right]^{1/2},$$

which yields

$$\int_0^1 |D^2 w|^2 \mathrm{d}z \geqslant \pi^2 \int_0^1 |\mathrm{D}w|^2 \mathrm{d}z, \tag{23}$$

while utilizing inequality (23), we obtain

$$\int_0^1 [|D^2 w|^2 + a^2 |Dw|^2] \mathrm{d}z \geqslant (\pi^2 + a^2) \int_0^1 |Dw|^2 \mathrm{d}z. \tag{24}$$

Also, since $pr \geqslant 0$, we get

$$\frac{pr}{\sigma} \int_0^1 [|Dw|^2 + a^2 |w|^2] \mathrm{d}z \geqslant 0. \tag{25}$$

Further, multiplying equation (2) throughout by θ^* (the complex conjugate θ), integrating each term of the resulting equation by parts for an appropriate number of times and making use of boundary condition on θ namely $\theta(0) = 0 = \theta(1)$, it follows that

$$\int_0^1 [|D\theta|^2 + a^2 |\theta|^2] \mathrm{d}z + pr \int_0^1 |\theta|^2 \mathrm{d}z = Re\left[\int_0^1 \theta^* w \mathrm{d}z \right],$$

$$\leqslant \left| \int_0^1 \theta^* w \mathrm{d}z \right|,$$

$$\leqslant \int_0^1 |\theta^*||w| \mathrm{d}z,$$

$$\leqslant \int_0^1 |\theta||w|\mathrm{d}z,$$

$$\left[\int_0^1 |\theta|^2\mathrm{d}z\right]^{1/2}\left[\int_0^1 |w|^2\mathrm{d}z\right]^{1/2}$$

(utilizing Schwartz inequality),

so that by using inequality (19) and the fact that $pr \geqslant 0$, we obtain from the above equation

$$(\pi^2+a^2)\left[\int_0^1 |\theta|^2\mathrm{d}z\right]^{1/2} \leqslant \left[\int_0^1 |w|^2\mathrm{d}z\right]^{1/2},$$

and hence

$$\int_0^1 [|D\theta|^2 + a^2|\theta|^2]\mathrm{d}z \leqslant \frac{1}{(\pi^2+a^2)}\int_0^1 |w|^2\mathrm{d}z \leqslant \frac{1}{\pi^2(\pi^2+a^2)}\int_0^1 |Dw|^2\mathrm{d}z,$$

(utilizing inequality 18) (26)

Also, multiplying equation (5) throughout by ζ^* (the complex conjugate of ζ), integrating each term of the resulting equation by parts for a suitable number of times and making use of boundary condition on ζ namely $\zeta(0)=0=\zeta(1)$, we get

$$\int_0^1 [|D\zeta|^2 + a^2|\zeta|^2]\mathrm{d}z + \frac{pr}{\sigma}\int_0^1 |\zeta|^2\mathrm{d}z = Re\left[\int_0^1 \zeta^* Dw\mathrm{d}z + Q\int_0^1 \zeta^* D\xi\mathrm{d}z\right]$$

$$= \mathrm{Re}\left[-\int_0^1 D\zeta^* w\mathrm{d}z - Q\int_0^1 D\zeta^*\xi\mathrm{d}z\right]$$

$$= \mathrm{Re}\left[-\int_0^1 D\zeta^* w\mathrm{d}z + Q\int_0^1 \xi\left(D^2 - a^2 - \frac{p^*\sigma_1}{\sigma}\right)\xi^*\mathrm{d}z\right],$$

(using equation (6)),

$$= \mathrm{Re}\left[-\int_0^1 D\zeta^* w\mathrm{d}z - Q\int_0^1 \left[|D\xi|^2 + a^2|\xi|^2 + \frac{p^*\sigma_1}{\sigma}|\xi|^2\mathrm{d}z\right]\right],$$

$$\leqslant \mathrm{Re}\left[-\int_0^1 D\zeta^* w\mathrm{d}z\right],$$

$$\leqslant \left|\int_0^1 D\zeta^* w\mathrm{d}z\right|,$$

$$\leqslant \int_0^1 |D\zeta^*||w|\mathrm{d}z,$$

$$\leqslant \int_0^1 |D\zeta||w|\mathrm{d}z,$$

$$\leqslant \left[\int_0^1 |D\zeta|^2\mathrm{d}z\right]^{1/2}\left[\int_0^1 |w|^2\mathrm{d}z\right]^{1/2}$$

(utilizing Schwartz inequality),

which implies, since $pr \geqslant 0$, that

$$\int_0^1 |D\zeta|^2 \mathrm{d}z \leqslant \int_0^1 |w|^2 \mathrm{d}z$$

so that using inequalities (18) and (21), we obtain from the above that

$$\int_0^1 |\zeta|^2 \mathrm{d}z \leqslant \frac{1}{\pi^4} \int_0^1 |Dw|^2 \mathrm{d}z. \tag{27}$$

Further, multiplying equation (4) throughout by h_z^* (the complex conjugate of h_z), integrating each term of the resulting equation by parts for a suitable number of times and making use of boundary condition on h_z namely $h_z(0) = 0 = h_z(1)$, we get

$$\int_0^1 [|Dhz|^2 + a^2|hz|^2] \mathrm{d}z + \frac{pr\sigma_1}{\sigma} \int_0^1 |hz|^2 \mathrm{d}z = \mathrm{Re}\left[\int_0^1 hz^* Dw\mathrm{d}z\right],$$

$$= \mathrm{Re}\left[-\int_0^1 Dhz^* w\mathrm{d}z\right],$$

$$\leqslant \left|-\int_0^1 Dhz^* w\mathrm{d}z\right|,$$

$$\leqslant \left|\int_0^1 Dhz^* w\mathrm{d}z\right|,$$

$$\leqslant \int_0^1 |Dhz^*||w| \mathrm{d}z,$$

$$\leqslant \int_0^1 |Dhz||w| \mathrm{d}z,$$

$$\leqslant \left[\int_0^1 |Dhz|^2 \mathrm{d}z\right]^{1/2} \left[\int_0^1 |w|^2 \mathrm{d}z\right]^{1/2}$$

(utilizing Schwartz inequality),

which implies, since $pr \geqslant 0$, that

$$\left[\int_0^1 |Dhz|^2 \mathrm{d}z\right]^{1/2} \leqslant \left[\int_0^1 |w|^2 \mathrm{d}z\right]^{1/2},$$

so that we obtain from the preceding equation that

$$\int_0^1 [|Dhz|^2 + a^2|hz|^2] \mathrm{d}z \leqslant \int_0^1 |w|^2 \mathrm{d}z,$$

which yields with the help of inequality (18) that

$$\int_0^1 [|Dhz|^2 + a^2|hz|^2] \mathrm{d}z \leqslant \frac{1}{\pi^2} \int_0^1 |Dw|^2 \mathrm{d}z. \tag{28}$$

Now, utilizing inequality (20), we have

$$\int_0^1 [|D\phi|^2 + a^2|\phi|^2]\mathrm{d}z \geqslant (\pi^2 + a^2) \int_0^1 |\phi|^2 \mathrm{d}z, \tag{29}$$

while from (16), we get

$$\int_0^1 |\phi|^2 \mathrm{d}z \geqslant \frac{1}{R_s a^2 \sigma} \int_0^1 |Dw|^2 \mathrm{d}z - \frac{T}{R_s a^2 \sigma} \int_0^1 |\zeta|^2 \mathrm{d}z - \frac{Q\sigma_1}{R_s a^2 \sigma} \int_0^1 [|Dh_z|^2 + a^2|h_z|^2]\mathrm{d}z, \tag{30}$$

so that using inequality (30), we can write the inequality (29) as

$$\int_0^1 [|D\phi|^2 + a^2|\phi|^2]\mathrm{d}z \geqslant \frac{(\pi^2 + a^2)}{R_s a^2 \sigma} \int_0^1 |Dw|^2 \mathrm{d}z - \frac{T(\pi^2 + a^2)}{R_s \sigma a^2} \int_0^1 |\zeta|^2 \mathrm{d}z - \frac{Q\sigma_1(\pi^2 + a^2)}{R_s a^2 \sigma} \int_0^1 [|Dhz|^2 + a^2|hz|^2]\mathrm{d}z. \tag{31}$$

Also, using inequalities (27) and (28), (31) yields

$$\int_0^1 [|D\phi|^2 + a^2|\phi|^2]\mathrm{d}z \geqslant \frac{(\pi^2 + a^2)}{R_s a^2 \sigma} \left[1 - \frac{Q\sigma_1}{\pi^2} - \frac{T}{\pi^4}\right] \int_0^1 |Dw|^2 \mathrm{d}z, \tag{32}$$

while equation (16) together with $pr \geqslant 0$ yields the inequality

$$pr\left[Ra^2 \int_0^1 |\theta|^2 \mathrm{d}z - R_s a^2 \int_0^1 |\phi|^2 dz - \frac{Q\sigma_1}{\sigma} \int_0^1 [|Dhz|^2 + a^2|hz|^2]\mathrm{d}z - \frac{T}{\sigma}\int_0^1 |\zeta|^2 \mathrm{d}z + \frac{QT\sigma_1}{\sigma} \int_0^1 |\xi|^2 \mathrm{d}z\right] \leqslant 0. \tag{33}$$

Now, if permissible let $R_s \geqslant R$. Then in that case we derive from equation (17) and inequalities (24), (26), (32) and (33) that

$$(\pi^2 + a^2) \int_0^1 |Dw|^2 \mathrm{d}z \leqslant \frac{R_s a^2}{\pi^2(\pi^2 + a^2)} \int_0^1 |Dw|^2 \mathrm{d}z - \frac{R_s a^2 \tau(\pi^2 + a^2)}{R_s a^2 \sigma} \left[1 - \frac{Q\sigma_1}{\pi^2} - \frac{T}{\pi^4}\right] \int_0^1 |Dw|^2 \mathrm{d}z, \tag{34}$$

which in turn yields that

$$\left[(\pi^2 + a^2)\left\{1 + \frac{\tau}{\sigma\pi^2}\left(\pi^2 - Q\sigma_1 - \frac{T}{\pi^2}\right)\right\} - \frac{R_s a^2}{\pi^2(\pi^2 + a^2)}\right] \int_0^1 |Dw|^2 \mathrm{d}z \leqslant 0 \tag{35}$$

which implies that

$$R_s > \frac{\pi^2(\pi^2+a^2)^2}{a^2}\left[1+\frac{\tau}{\sigma\pi^2}\left\{\pi^2-(Q\sigma_1+\frac{T}{\pi^2}\right\}\right], \tag{36}$$

and thus we must necessarily have

$$R_s > 4\pi^4\left[1+\frac{\tau}{\sigma\pi^2}\left\{\pi^2-\left(Q\sigma_1+\frac{T}{\pi^2}\right)\right\}\right], \tag{37}$$

since the minimum value of $\frac{\pi^2(\pi^2+a^2)^2}{a^2}$ is $4\pi^4$ for $a^2=\pi^2>0$.

Hence, if $R_s \leqslant 4\pi^4\left[1+\frac{\tau}{\sigma\pi^2}\left\{\pi^2-\left(Q\sigma_1+\frac{T}{\pi^2}\right)\right\}\right]$, then we must have

$$R_s < R, \tag{38}$$

and this completes the proof of the theorem.

This result is analogous to the corresponding result of Banerjee *et al.* [2] in hydrodynamic thermohaline convection of Veronis [7] type and can be stated as follows in an equivalent form namely, "Magnetorotatory hydrodynamic thermohaline convection of the Veronis [7] type cannot manifest as oscillatory motions of growing amplitude in an initially bottom heavy configuration if R_s, τ, σ, σ_1, Q T satisfy the inequality

$$R_s \leqslant 4\pi^4\left[1+\frac{\tau}{\sigma\pi^2}\left\{\pi^2-\left(Q\sigma_1+\frac{T}{\pi^2}\right)\right\}\right],$$

when both the boundary surfaces are rigid". It may be noted that for the hydrodynamic case in which $T=0$ and $Q=0$, we recover the result due to Banerjee *et al* [2] from Theorem 1.

Theorem 2. *If* $R<0$, $R_s<0$, $T>0$, $Q>0$, $pr \geqslant 0$, $pi \neq 0$, $\frac{Q\sigma_1}{\pi^2}+\frac{T}{\pi^4} \leqslant (1+\sigma)$ *and* $|R| \leqslant 4\pi^4\tau\left[1+\frac{1}{\sigma\pi^2}\left\{\pi^2-\left(Q\sigma_1+\frac{T}{\pi^2}\right)\right\}\right.$ *then a necessary condition for the existence of a non-trivial solution* $(w,\theta,\phi,h_z,\zeta,\xi,p)$ *of equations* (1)–(6) *together with boundary conditions* (7) is

$$|R| < |R_s|.$$

The validity of the theorem is established by following a procedure similar to the one adopted proving Theorem 1.

The above theorem can be stated as follows in an equivalent form namely, "Magnetorotatory thermohaline convection of the Stern (1960) type cannot manifest itself as oscillatory motions of growing amplitude in an initially bottom heavy configuration if R, τ, σ, σ_1, Q and T satisfy the inequality

$$|R| \leqslant 4\pi^4\left[1+\frac{1}{\sigma\pi^2}\left\{\pi^2-\left(Q\sigma_1+\frac{T}{\pi^2}\right)\right\}\right],$$

when both the bounding surfaces are rigid.

Further, it may be noted that for the corresponding hydrodynamic case in which $T = 0$ and $Q = 0$, we recover the results due to Jyoti Prakash (1992) from Theorem 2.

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Gysin maps and cycle classes for Hodge cohomology

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Abstract. If $f:X \to Y$ is a projective morphism between regular varieties over a field, we construct Gysin maps

$$f_*:H^i(X,\Omega^j_{X/\mathbf{Z}}) \to H^{i+d}_{f(X)}(Y,\Omega^{j+d}_{Y/\mathbf{Z}})$$

for the Hodge cohomology groups, where $d = \dim Y - \dim X$. These Gysin maps have the expected properties, and in particular may be used to construct a cycle class map

$$Cl_X:CH^i(X,S) \to H^i(X,\Omega^i_{X/\mathbf{Z}}),$$

where X is quasi-projective over a field, S is the singular locus, and $CH^i(X,S)$ is the relative Chow group of codimension-i cycles modulo rational equivalence. Simple properties of this cycle map easily imply the infinite dimensionality theorem for the Chow group of zero cycles of a normal projective variety X over $\mathbf{C}$ with $H^n(X,\mathcal{O}_X) \neq 0$, where $n = \dim X$. One also recovers examples of Nori of affine n-dimensional varieties which support indecomposable vector bundles of rank n.

Keywords. Gysin maps; Hodge cohomology; cycle classes.

Let X be a scheme. Let $\Omega^1_{X/\mathbf{Z}}$ be the sheaf of absolute Kähler differentials on X, and let $\Omega^i_{X/\mathbf{Z}} = \overset{i}{\wedge}_{\mathcal{O}_X}(\Omega^1_{X/\mathbf{Z}})$. The *Hodge cohomology groups* are the groups $H^i(X,\Omega^j_{X/\mathbf{Z}})$.

Let $f:X \to Y$ be a projective morphism between integral regular varieties, which are of finite type over a field. Let $d = \dim Y - \dim X$. We construct (see Theorem 1) Gysin homomorphisms

$$f_*:H^i(X,\Omega^j_{X/\mathbf{Z}}) \to H^{i+d}_{f(X)}(Y,\Omega^{j+d}_{Y/\mathbf{Z}})$$

where $H^r_{f(X)}(Y,\Omega^s_{Y/\mathbf{Z}})$ denotes the (Hodge) cohomology with support in the Zariski closed subset $f(X)$ (we define this to be 0 if either r or s is negative). We prove that these Gysin maps have the expected properties (compatibility with compositions, projection formula, compatibility with flat base change, etc.). This is done in § 1, and occupies the bulk of the paper. The proofs are fairly elementary, but somewhat tedious.

In § 2, we use the Gysin maps to construct cycle class homomorphisms

$$Cl^i_X:CH^i(X,S) \to H^i(X,\Omega^i_{X/\mathbf{Z}}),$$

where X is an integral separated k-scheme of finite type, and S is the locus of non-regular points (i.e., the singular locus). Here $CH^i(X,S)$ is the relative Chow group of rational equivalence classes of cycles of codimension i on X supported in the regular locus.

In § 3, we give two applications of the cycle class. One is really an application of the existence of the standard cycle class in $H^i(X, \Omega^i_{X/k})$, but does not seem to have been explicitly noted in the literature in the case when k has positive characteristic. Next, the refined cycle class constructed above is used to give a "natural" proof of the infinite dimensionality theorem for the Chow group of zero cycles on a normal projective variety over $\mathbf{C}$. This was first proved for smooth surfaces by Mumford [M], and for smooth X of any dimension by Roitman [R]; for normal X this was proved in [S]. All these proofs follow Mumford's proof for surfaces, which is somewhat indirect. Bloch [B] proved the infinite dimensionality theorem for smooth X using correspondences, and the "generic zero cycle" (essentially, the diagonal on $X \times X$). This work arose out of our attempts to give an analogous proof for normal X, where Bloch's argument, as it stands, breaks down; instead of using correspondences, we use properties of the refined cycle class map.

1. Gysin maps

We construct Gysin maps for the Hodge cohomology groups in the following context. A proper morphism $f: X \to Y$ will be called *admissible* if

(i) X and Y are Noetherian integral separated by regular schemes,
(ii) $\Omega^1_{X/\mathbf{Z}}, \Omega^1_{Y/\mathbf{Z}}$ are locally free sheaves on X, Y respectively (not necessarily coherent),
(iii) there is a locally free sheaf $\mathscr{E}$ of finite rank on Y and a morphism $i: X \to \mathbf{P}_Y(\mathscr{E})$ which is a closed embedding of Y-schemes (since Y is regular, this is equivalent of f being *projective* in the sense of E.G.A. II).

(We recall that an $\mathcal{O}_X$-module $\mathscr{E}$ is locally free if X is covered by affine open subsets U_i such that $\mathscr{E}|_{U_i}$ is free, for all i; this is *not* equivalent to the condition that the stalks of $\mathscr{E}$ are all free modules, in general. This is true, however, if $\mathscr{E}$ is the direct sum of a locally free $\mathcal{O}_X$-module and a coherent $\mathcal{O}_X$-module, or even if this holds locally on X.)

We will construct Gysin maps for Hodge cohomology for admissible morphisms $f: X \to Y$.

In fact, if S is a base scheme, one can imitate the arguments given here to construct Gysin maps

$$f_*: H^p(X, \Omega^q_{X/S}) \to H^{p+d}_{f(X)}(Y, \Omega^{q+d}_{Y/S})$$

for S-admissible morphisms (projective morphisms between Noetherian integral regular S-schemes X, Y such that $\Omega^1_{X/S}$, $\Omega^1_{Y/S}$ are locally free). We have not done this here since we do not have any application of this more general Gysin map.

The next lemma and its corollary imply that any projective morphism between regular varieties over a field is admissible.

Lemma 1. *Let R be a regular local ring which is the localisation of a finitely generated algebra over a field k. Then $\Omega^1_{R/\mathbf{Z}}$ is a free R-module.*

Proof. If k has characteristic zero, then $\Omega^1_{R/k}$ is a free R-module, since R is the local ring of a smooth point on a variety over k. There is an exact sequence

$$\Omega^1_{k/\mathbf{Z}} \otimes_k R \xrightarrow{\varepsilon} \Omega^1_{R/\mathbf{Z}} \to \Omega^1_{R/k} \to 0,$$

where $\Omega^1_{k/\mathbf{Z}} \otimes_k R$ is a free R-module. So it suffices to note that ε is injective—indeed, if K is the quotient field of R, then

$$\Omega^1_{k/\mathbf{Z}} \otimes_k K \to \Omega^1_{K/\mathbf{Z}}$$

is injective, since the latter has a K-basis given by $\{dt_\alpha\}$ where $\{t_\alpha\}$ is a transcendence basis for K over $\mathbf{Q}$, and we may choose $\{t_\alpha\}$ to contain a transcendence basis for k over $\mathbf{Q}$.

If k has positive characteristic, then this is corollary (3.6) of [KN]. (See also [K], Chapter 7). □

COROLLARY 1

Let X be an integral regular variety over a field. Then $\Omega^1_{X/\mathbf{Z}}$ is a locally free $\mathcal{O}_X$-module.

Proof. It suffices to prove this when $X = \operatorname{Spec} A$ is affine, where A is a regular finitely generated algebra over a field k. Now the module of absolute Kähler differentials of a polynomial algebra over k is a free module. We may write $A = k[X_1, \ldots, X_n]/I$, a quotient of a polynomial algebra over k, so that $\Omega^1_{A/\mathbf{Z}}$ has a presentation

$$I/I^2 \to \Omega^1_{k[X_1,\ldots,X_n]/\mathbf{Z}} \otimes A \to \Omega^1_{A/\mathbf{Z}} \to 0.$$

Thus $\Omega^1_{A/\mathbf{Z}}$ is a quotient of a free A-module by a finitely generated submodule. Hence $\Omega^1_{A/\mathbf{Z}}$ can be expressed as a direct sum of a free A-module and a finitely generated A-module. From the above lemma, $\Omega^1_{A/\mathbf{Z}}$ is free when localized at any prime ideal of A; hence $\Omega^1_{A/\mathbf{Z}}$ is the direct sum of a free A-module and a finitely generated projective A-module. □

We first construct Gysin maps for admissible closed embeddings. Let $i: X \to Y$ be an admissible closed embedding, and let $d = \dim Y - \dim X$ ($\geqslant 0$); the Gysin maps are maps

$$i_*: H^p(X, \Omega^q_{X/\mathbf{Z}}) \to H^{p+d}_{i(X)}(Y, \Omega^{q+d}_{Y/\mathbf{Z}}).$$

Let $\mathcal{I}$ be the ideal sheaf of $i(X)$ in Y. Since $\mathcal{I}$ is locally generated by a regular sequence, the corresponding Koszul complex yields a natural isomorphism of locally free $\mathcal{O}_X$-modules

$$i^* \mathcal{E}xt^d_Y(i_* \mathcal{O}_X, \mathcal{O}_Y) \cong \mathcal{H}om_{\mathcal{O}_X}(i^* \overset{d}{\wedge} (\mathcal{I}/\mathcal{I}^2), \mathcal{O}_X),$$

and

$$\mathcal{E}xt^s_Y(i_* \mathcal{O}_X, \mathcal{O}_Y) = 0 \quad \forall s \neq d.$$

If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module, and $\mathcal{F}$ is a locally free $\mathcal{O}_Y$-module, then we have an induced isomorphism

$$i^* \mathcal{E}xt^d_Y(i_* \mathcal{E}, \mathcal{F}) \cong \mathcal{H}om_{\mathcal{O}_X}(i^* \overset{d}{\wedge} (\mathcal{I}/\mathcal{I}^2) \otimes \mathcal{E}, i^* \mathcal{F}),$$

natural in $\mathscr{E}$, $\mathscr{F}$ in an obvious sense, and

$$\mathscr{E}xt^s_Y(i_*\mathscr{E},\mathscr{F})=0 \quad \forall s\neq d.$$

This is true even if $\mathscr{E}$, $\mathscr{F}$ are not coherent; this is because $\mathscr{E}xt$ is the same whether it is computed in the categories of quasi-coherent $\mathcal{O}_Y$-modules, or in the category of all $\mathcal{O}_Y$-modules (for each fixed $\mathscr{E}$, we regard both as derived functors of the appropriate $\mathscr{H}om(\mathscr{E},-)$ with values in the category of all $\mathcal{O}_Y$-modules). This follows from the fact that any quasi-coherent $\mathcal{O}_Y$-module is an $\mathcal{O}_Y$-submodule of an injective $\mathcal{O}_Y$-module which is quasi-coherent (see [H2], II, § 7). Now on an affine open set Spec $A\subset Y$, on which the ideal $\Gamma(U,\mathscr{I})\subset A$ is generated by a regular sequence, and $\mathscr{E}$, $\mathscr{F}$ are free, we may compute the $\mathscr{E}xt$ in the category of quasi-coherent $\mathcal{O}_Y$-modules using a Koszul complex.

In particular, since i is admissible, $\Omega^q_{X/Z}$ is a locally free $\mathcal{O}_X$-module, and $\Omega^{q+d}_{Y/Z}$ is a locally free $\mathcal{O}_Y$-module, so that

$$i^*\mathscr{E}xt^d_Y(i_*\Omega^q_{X/Z},\Omega^{q+d}_{Y/Z})\cong\mathscr{H}om_{\mathcal{O}_X}\Big(i^*\overset{d}{\wedge}(\mathscr{I}/\mathscr{I}^2)\otimes\Omega^q_{X/Z},i^*\Omega^{q+d}_{Y/Z}\Big),$$

and

$$\mathscr{E}xt^s_Y(i_*\Omega^q_{X/Z},\Omega^{q+d}_{Y/Z})=0\quad\forall s\neq d.$$

From the spectral sequence

$$E_2^{r,s}=H^r(X,\mathscr{E}xt^s_Y(i_*\Omega^q_{X/Z},\Omega^{q+d}_{Y/Z}))\Rightarrow\mathrm{Ext}^{r+s}_Y(i_*\Omega^q_{X/Z},\Omega^{q+d}_{Y/Z})$$

we obtain

$$\mathrm{Ext}^d_Y(i_*\Omega^q_{X/Z},\Omega^{q+d}_{Y/Z})\cong\mathrm{Hom}_X\Big(i^*\overset{d}{\wedge}(\mathscr{I}/\mathscr{I}^2)\otimes\Omega^q_{X/Z},i^*\Omega^{q+d}_{Y/Z}\Big).$$

From the exact sequence of locally free $\mathcal{O}_X$-modules

$$0\to i^*\mathscr{I}/\mathscr{I}^2\to i^*\Omega^1_{Y/Z}\to\Omega^1_{X/Z}\to 0$$

we obtain maps

$$i^*\overset{d}{\wedge}(\mathscr{I}/\mathscr{I}^2)\otimes\Omega^q_{X/Z}\to i^*\Omega^{q+d}_{Y/Z},$$

and hence elements

$$\alpha_q(i)\in\mathrm{Ext}^d_Y(i_*\Omega^q_{X/Z},\Omega^{q+d}_{Y/Z}).$$

The Gysin map

$$i_*:H^p(X,\Omega^q_{X/Z})\to H^{p+d}_{i(X)}(Y,\Omega^{q+d}_{Y/Z})$$

is essentially the Yoneda product with $\alpha_q(i)$. More precisely, we have a map

$$\theta:\mathrm{Ext}^{p+d}_Y(i_*\mathcal{O}_X,\Omega^{q+d}_{Y/Z})\to H^{p+d}_{i(X)}(Y,\Omega^{q+d}_{Y/Z})$$

induced by the isomorphism (see [G2])

$$\varinjlim_m\mathrm{Ext}^{p+d}_Y(\mathcal{O}_Y/\mathscr{I}^m,\Omega^{q+d}_{Y/Z})\cong H^{p+d}_{i(X)}(Y,\Omega^{q+d}_{Y/Z}).$$

The Gysin map is the composite

$$\begin{aligned} H^p(X,\Omega^q_{X/Z}) &\cong \operatorname{Ext}^p_X(\mathcal{O}_X,\Omega^q_{X/Z}) \xrightarrow{i_*} \operatorname{Ext}^p_Y(i_*\mathcal{O}_X, i_*\Omega^q_{X/Z}) \\ &\xrightarrow{\cdot\alpha_q(i)} \operatorname{Ext}^{p+d}_Y(i_*\mathcal{O}_X,\Omega^{q+d}_{Y/Z}) \xrightarrow{\theta} H^{p+d}_{i(X)}(Y,\Omega^{q+d}_{Y/Z}). \end{aligned}$$

Here $\cdot\alpha_q(i)$ denotes the Yoneda product with $\alpha_q(i)$.

If $i:X\to Y$ is a closed embedding, we often write $H^r_X(Y,-)$ instead of $H^r_{i(X)}(Y,-)$, to simplify notation.

Lemma 2. *Suppose $i:X\to Y, j:Y\to Z$ are admissible closed embeddings. Let $d=\dim Y-\dim X$, $e=\dim Z-\dim Y$. Let*

$$\begin{aligned} &\psi:H^{p+q}_X(Y,\Omega^{q+d}_{Y/Z})\to H^{p+d}(Y,\Omega^{q+d}_{Y/Z}), \\ &\theta:H^{p+d+e}_X(Z,\Omega^{q+d+e}_{Z/Z})\to H^{p+d+e}_Y(Z,\Omega^{q+d+e}_{Z/Z}) \end{aligned}$$

be the natural maps. Then

$$j_*\circ\psi\circ i_*=\theta\circ(j\circ i)_*:H^p(X,\Omega^q_{X/Z})\to H^{p+d+e}_Y(Z,\Omega^{q+d+e}_{Z/Z}).$$

Proof. Let $\tilde{\alpha}_q(i)$ be the image of $\alpha_q(i)$ under the map

$$j_*:\operatorname{Ext}^d_Y(i_*\Omega^d_{X/Z},\Omega^{q+d}_{Y/Z})\to\operatorname{Ext}^d_Z((j\circ i)_*\Omega^q_{X/Z},j_*\Omega^{q+d}_{Y/Z}).$$

One easily reduces to showing that

$$\alpha_q(j\circ i)\in\operatorname{Ext}^{d+e}_Z((j\circ i)_*\Omega^q_{X/Z},\Omega^{q+d+e}_{Z/Z})$$

is the Yoneda product $\tilde{\alpha}_q(i)\cdot\alpha_{q+d}(j)$. Now $\alpha_q(j\circ i)$, $\tilde{\alpha}_q(i)$ and $\alpha_{q+d}(j)$ yield global sections (which we denote by the same symbols) of the corresponding $\mathcal{E}xt$ sheaves, and $\alpha_q(j\circ i)$, $\alpha_{q+d}(j)$ are determined uniquely by these global sections. Hence it suffices to show that the sheaf theoretic Yoneda product of $\alpha_{q+d}(j)$ and $\tilde{\alpha}_q(i)$ is $\alpha_q(j\circ i)$.

Let $\mathcal{I}$ be the ideal sheaf of $(j\circ i)(X)\subset Z$, and let $\mathcal{J}\subset\mathcal{I}$ be the ideal sheaf of $j(Y)\subset Z$. The composite pairing

$$\begin{tikzcd}[column sep=small] j_*\mathcal{E}xt^d_Y(i_*\Omega^q_{X/Z},\Omega^{q+d}_{Y/Z}) & \otimes \arrow[d] & \mathcal{E}xt^e_Z(j_*\Omega^{q+d}_{Y/Z},\Omega^{q+d+e}_{Z/Z}) \\ \mathcal{E}xt^d_Z((j\circ i)_*\Omega^q_{X/Z},j_*\Omega^{q+d}_{Y/Z}) & \otimes \arrow[d, "\text{Yoneda}"] & \mathcal{E}xt^e_Z(j_*\Omega^{q+d}_{Y/Z},\Omega^{q+d+e}_{Z/Z}) \\ & \mathcal{E}xt^{d+e}_Z((j\circ i)_*\Omega^q_{X/Z},\Omega^{q+d+e}_{Z/Z}) & \end{tikzcd}$$

may be viewed as a pairing (denoted by ψ)

$$\begin{aligned} &(j\circ i)_*\mathcal{H}om_X\Big((j\circ i)^*\overset{d}{\wedge}(\mathcal{I}/\mathcal{I}^2+\mathcal{J})\otimes\Omega^q_{X/Z},i^*\Omega^{q+d}_{Y/Z}\Big)\otimes \\ &\quad j_*\mathcal{H}om_Y\Big(j^*\overset{e}{\wedge}(\mathcal{J}/\mathcal{J}^2)\otimes\Omega^{q+d}_{Y/Z},j^*\Omega^{q+d+e}_{Z/Z}\Big) \\ &\quad\xrightarrow{\psi}(j\circ i)_*\mathcal{H}om_X\Big((j\circ i)^*\overset{d+e}{\wedge}(\mathcal{I}/\mathcal{I}^2)\otimes\Omega^q_{X/Z},(j\circ i)^*\Omega^{q+d+e}_{Z/Z}\Big). \end{aligned}$$

On the other hand, from the exact sequence of locally free $\mathcal{O}_X$-modules

$$0\to (j\circ i)^*(\mathscr{J}/\mathscr{J}^2)\to (j\circ i)^*(\mathscr{I}/\mathscr{I}^2)\to (j\circ i)^*(\mathscr{I}/\mathscr{I}^2+\mathscr{J})\to 0$$

we have an isomorphism

$$(j\circ i)^* \overset{d+e}{\wedge}(\mathscr{I}/\mathscr{I}^2)\cong (j\circ i)^* \overset{d}{\wedge}(\mathscr{I}/\mathscr{I}^2+\mathscr{J})\otimes (j\circ i)^* \overset{e}{\wedge}(\mathscr{J}/\mathscr{J}^2).$$

This yields a pairing (denoted η)

$$(j\circ i)_* \mathscr{H}om_X((j\circ i)^* \overset{d}{\wedge}(\mathscr{I}/\mathscr{I}^2+\mathscr{J}),\mathcal{O}_X)\otimes j_*\mathscr{H}om_Y(j^* \overset{e}{\wedge}(\mathscr{J}/\mathscr{J}^2),\mathcal{O}_Y)$$
$$\xrightarrow{\eta}(j\circ i)_*\mathscr{H}om((j\circ i)^* \overset{d+e}{\wedge}(\mathscr{I}/\mathscr{I}^2),\mathcal{O}_X),$$

and hence yields another pairing ψ' with the same domain and range as ψ; an easy computation shows that

$$\psi'(\tilde{\alpha}_q(i)\otimes \alpha_{q+d}(j))=\alpha_q(j\circ i).$$

Hence it suffices to prove that $\psi=\psi'$. This amounts to showing that η is (equivalent with) the following composite pairing:

$$\begin{tikzcd}
j_*\mathscr{E}xt^d_Y(i_*\mathcal{O}_X,\mathcal{O}_Y) & \otimes_{\mathcal{O}_Z} \arrow[d] & \mathscr{E}xt^e(j_*\mathcal{O}_Y,\mathcal{O}_Z) \\
\mathscr{E}xt^d_Z((j\circ i)_*\mathcal{O}_X,j_*\mathcal{O}_Y) & \otimes_{\mathcal{O}_Z} \arrow[d, "\text{Yoneda}"] & \mathscr{E}xt^e_Z(j_*\mathcal{O}_Y,\mathcal{O}_Z) \\
 & \mathscr{E}xt^{d+e}_Z((j\circ i)_*\mathcal{O}_X,\mathcal{O}_Z) &
\end{tikzcd}$$

We may check this locally on Z. Let $Z=\operatorname{Spec} A$, $X=\operatorname{Spec} A/I$, $Y=\operatorname{Spec} A/J$ where $J\subset I\subset A$ are ideals generated by regular sequences. We may choose generators $f_1,\dots,f_d,g_1,\dots,g_e$ of I, forming a regular sequence, such that $g_1,\dots,g_e\in J$ are a system of generators, and the images $\bar{f}_i$ of f_i in A/J yield a regular sequence generating I/J. Then

$$\overset{d}{\wedge}(I/I^2+J)=\bar{f}_1\wedge\cdots\wedge\bar{f}_d A/I\cong A/I,$$

and

$$\operatorname{Ext}^d_{A/J}(A/I,A/J)\cong \operatorname{Hom}_{A/I}(\overset{d}{\wedge}(I/I^2+J),A/I)\cong A/I,$$

where $1\in A/I$ corresponds to the class of the d-extension

$$0\to \overset{d}{\wedge}((A/J)^{\oplus d})\to\cdots\to \overset{2}{\wedge}((A/J)^{\oplus d})\to (A/J)^{\oplus d}\to A/J\to A/I\to 0$$

given by the Koszul complex $K^{\cdot}_{A/J}(\bar{f}_1,\dots,\bar{f}_d)$. Similarly

$$\overset{e}{\wedge}(J/J^2)=g_1\wedge\cdots\wedge g_e A/J\cong A/J$$

and

$$\operatorname{Ext}^e_A(A/J,A)\cong \operatorname{Hom}_{A/J}(\overset{e}{\wedge}(J/J^2),A/J)\cong A/J$$

with $1 \in A/J$ corresponding to the Koszul complex $K^{\cdot}_A(g_1,\ldots,g_e)$,

$$0 \to \overset{e}{\wedge}(A^{\oplus e}) \to \cdots \to \overset{2}{\wedge}(A^{\oplus e}) \to A^{\oplus e} \to A \to A/J \to 0.$$

Finally

$$\overset{d+e}{\wedge}(I/I^2) = f_1 \wedge \cdots \wedge f_d \wedge g_1 \wedge \cdots \wedge g_e A/I \cong A/I,$$

and

$$\operatorname{Ext}^{d+e}_A(A/I, A) \cong \operatorname{Hom}_{A/I}(\overset{d+e}{\wedge}(I/I^2), A/I) \cong A/I,$$

where $1 \in A/I$ corresponds to the Koszul complex $K^{\cdot}_A(f_1,\ldots,f_d,g_1,\ldots,g_e)$,

$$0 \to \overset{d+e}{\wedge} A^{\oplus d+e} \to \cdots \to \overset{2}{\wedge} A^{\oplus d+e} \to A^{\oplus d+e} \to A \to A/I \to 0.$$

Since the isomorphism (used to define η)

$$\overset{d+e}{\wedge}(I/I^2) \cong \overset{d}{\wedge}(I/I^2 + J) \otimes \overset{e}{\wedge}(J/J^2)$$

satisfies

$$f_1 \wedge \cdots \wedge f_d \wedge g_1 \wedge \cdots \wedge g_e \mapsto (\bar{f}_1 \wedge \cdots \wedge \bar{f}_d) \otimes (g_1 \wedge \cdots \wedge g_e),$$

we are reduced to showing that the composite pairing

$$\operatorname{Ext}^d_{A/J}(A/I, A/J) \otimes_A \operatorname{Ext}^e_A(A/J, A) \to \operatorname{Ext}^d_A(A/I, A/J) \otimes_A \operatorname{Ext}^e_A(A/J, A)$$
$$\xrightarrow{\text{Yoneda}} \operatorname{Ext}^{d+e}_A(A/I, A)$$

is just

$$A/I \otimes_A A/J \to A/I, 1 \otimes 1 \mapsto 1.$$

The Yoneda product of the two Koszul complexes $K^{\cdot}_{A/J}(\bar{f}_1,\ldots,\bar{f}_d)$, $K^{\cdot}_A(g_1,\ldots,g_e)$ is the $(d+e)$-extension

$$0 \to \overset{e}{\wedge}(A^{\oplus e}) \to \cdots \to \overset{2}{\wedge}(A^{\oplus e}) \to A^{\oplus e} \to A \xrightarrow{\mu} \overset{d-1}{\wedge}((A/J)^{\oplus d}) \to$$
$$\cdots \to (A/J)^{\oplus d} \to A/J \to A/I \to 0.$$

Here μ is the composite

$$A \to A/J \cong \overset{d}{\wedge}((A/J)^{\oplus d}) \to \overset{d-1}{\wedge}((A/J)^{\oplus d})$$

obtained by splicing the two complexes.

There is a map of $(d+e)$-extensions from the Koszul complex $K^{\cdot}_A(f_1,\ldots,f_d,g_1,\ldots,g_e)$ onto the above $(d+e)$-extension, defined as follows. The split exact sequence

$$0 \to A^{\oplus e} \to A^{\oplus d+e} \to A^{\oplus d} \to 0$$

yields surjections

$$\alpha_j\colon \overset{j}{\wedge} A^{\oplus d+e} \to \overset{j}{\wedge} A^{\oplus d} \to \overset{j}{\wedge}((A/J)^{\oplus d}), \quad 0 \leqslant j \leqslant d-1.$$

We also have surjections

$$\alpha_j \colon \overset{j}{\wedge} A^{\oplus d+e} \to \big(\overset{j-d}{\wedge} A^{\oplus e}\big) \otimes \overset{d}{\wedge} A^{\oplus d} \cong \big(\overset{j-d}{\wedge} A^{\oplus e}\big), \quad d \leqslant j \leqslant d+e.$$

where if $x_1, \ldots, x_d$ is the standard basis for $A^{\oplus d}$ then $x_1 \wedge \cdots \wedge x_d \in \overset{d}{\wedge} A^{\oplus d}$ is the generator chosen to obtain the last isomorphism. The maps $\alpha_j, 0 \leqslant j \leqslant d+e$, give a map of $(d+e)$-extensions, since one easily computes that they are compatible with the differentials. Hence the two $(d+e)$-extensions represent the same class in $\mathrm{Ext}^{d+e}_A(A/I, A)$. □

Lemma 3. *Let*

$$\begin{tikzcd} X' \arrow[r, "i'"] \arrow[d, "g"'] & Y' \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

be a fibre square where i, i′ are admissible closed embeddings, and f, g are flat. Let $d = \dim Y - \dim X = \dim Y' - \dim X'$. *Then for all* $p, q \geqslant 0$, *the following diagram commutes*:

$$\begin{tikzcd} H^p(X', \Omega^q_{X'/\mathbf{Z}}) \arrow[r, "i'_*"] & H^{p+d}_{X'}(Y', \Omega^{q+d}_{Y'/\mathbf{Z}}) \\ H^p(X, \Omega^q_{X/\mathbf{Z}}) \arrow[r, "i_*"] \arrow[u, "g^*"] & H^{p+d}_X(Y, \Omega^{q+d}_{Y/\mathbf{Z}}) \arrow[u, "f^*"'] \end{tikzcd}$$

where f^*, g^* *are the natural maps.*

Proof. Let $g^*\alpha_q(i)$ be the image of $\alpha_q(i)$ under the map

$$g^* \colon \mathrm{Ext}^d_Y(i_*\Omega^q_{X/\mathbf{Z}}, \Omega^{q+d}_{Y/\mathbf{Z}}) \to \mathrm{Ext}^d_{Y'}(i'_* f^*\Omega^q_{X/\mathbf{Z}}, g^*\Omega^{q+d}_{Y/\mathbf{Z}}).$$

The proof of the lemma is easily reduced to the commutativity of the square

$$\begin{tikzcd} \mathrm{Ext}^p_{Y'}(i'_*\mathcal{O}_{X'}, i'_* f^*\Omega^q_{X/\mathbf{Z}}) \arrow[r, "\cdot g^*\alpha_q(i)"] \arrow[d, "\varphi_1"'] & \mathrm{Ext}^{p+d}_{Y'}(i'_*\mathcal{O}_{X'}, g^*\Omega^{q+d}_{Y/\mathbf{Z}}) \arrow[d, "\varphi_2"] \\ \mathrm{Ext}^p_{Y'}(i'_*\mathcal{O}_{X'}, i'_*\Omega^q_{X'/\mathbf{Z}}) \arrow[r, "\cdot\alpha_q(i')"] & \mathrm{Ext}^{p+d}_{Y'}(i'_*\mathcal{O}_{X'}, \Omega^{q+d}_{Y'/\mathbf{Z}}) \end{tikzcd}$$

where the horizontal maps are Yoneda products, and the vertical maps are induced by the natural maps

$$f^*\Omega^q_{X/\mathbf{Z}} \to \Omega^q_{X'/\mathbf{Z}}, g^*\Omega^{q+d}_{Y/\mathbf{Z}} \to \Omega^{q+d}_{Y'/\mathbf{Z}}.$$

Let β_1 be the image of $g^*\alpha_q(i)$ under

$$\mathrm{Ext}^d_{Y'}(i'_* f^*\Omega^q_{X/\mathbf{Z}}, g^*\Omega^{q+d}_{Y/\mathbf{Z}}) \to \mathrm{Ext}^d_{Y'}(i'_* f^*\Omega^q_{X/\mathbf{Z}}, \Omega^{q+d}_{Y'/\mathbf{Z}}),$$

and let β_2 be the image of $\alpha_q(i')$ under

$$\mathrm{Ext}^d_{Y'}(i'_*\Omega^q_{X'/\mathbf{Z}}, \Omega^{q+d}_{Y'/\mathbf{Z}}) \to \mathrm{Ext}^d_{Y'}(i'_* f^*\Omega^q_{X/\mathbf{Z}}, \Omega^{q+d}_{Y'/\mathbf{Z}}).$$

Then $\varphi_2 \circ (\cdot g^*\alpha_q(i))$ is the Yoneda product with β_1, while $(\cdot\alpha(i'))\circ\varphi_1$ is the Yoneda product with β_2. Hence it suffices to prove that $\beta_1 = \beta_2$.

Now $\beta_1, \beta_2, g^*\alpha_q(i), \alpha_q(i')$ all yield global sections (which we denote by the same symbols) of the corresponding $\mathscr{E}xt$ sheaves, which are in turn isomorphic to certain $\mathscr{H}om$ sheaves.

Let $\mathscr{I}$ be the ideal sheaf of $i(X)$ in Y; then $f^*\mathscr{I}$ is the ideal sheaf of $X' \subset Y'$. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & f^*i^*\mathscr{I}/\mathscr{I}^2 \arrow[r] \arrow[d, equal] & i'^*g^*\Omega^1_{Y/Z} \arrow[r] \arrow[d] & f^*\Omega^1_{X/Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f^*i^*\mathscr{I}/\mathscr{I}^2 \arrow[r] & i'^*\Omega^1_{Y'/Z} \arrow[r] & \Omega^1_{X'/Z} \arrow[r] & 0
\end{tikzcd}$$

which yields a diagram

$$\begin{tikzcd}
f^*i^*\bigwedge^d(\mathscr{I}/\mathscr{I}^2)\otimes f^*\Omega^q_{X/Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & i'^*g^*\Omega^{q+d}_{Y/Z} \arrow[d] \\
f^*i^*\bigwedge^d(\mathscr{I}/\mathscr{I}^2)\otimes\Omega^q_{X'/Z} \arrow[r] & i'^*\Omega^{q+d}_{Y'/Z}
\end{tikzcd}$$

where the upper horizontal arrow corresponds to

$$\begin{aligned}
g^*\alpha_q(i) &\in \Gamma(Y', i'_*\mathscr{H}om_{X'}(f^*i^*\textstyle\bigwedge^d(\mathscr{I}/\mathscr{I}^2)\otimes f^*\Omega^q_{X/Z}, i'^*g^*\Omega^{q+d}_{Y/Z}))\\
&\cong \operatorname{Ext}^d_{Y'}(f^*\Omega^q_{X/Z}, g^*\Omega^{q+d}_{Y/Z}),
\end{aligned}$$

and the lower horizontal arrow similarly corresponds to $\alpha_q(i')$. From the naturality properties of the isomorphism

$$i'^*\mathscr{E}xt^d_{Y'}(i'_*f^*\mathscr{E},\mathscr{F}) \cong \mathscr{H}om_{X'}(f^*i^*\textstyle\bigwedge^d(\mathscr{I}/\mathscr{I}^2)\otimes f^*\mathscr{E}, i'^*\mathscr{F})$$

(for $\mathscr{E}$ a locally free $\mathscr{O}_X$-module, and $\mathscr{F}$ a locally free $\mathscr{O}_{Y'}$-module), we see that the two (equal) composite maps (given by the diagram $\diamond$)

$$f^*i^*\textstyle\bigwedge^d(\mathscr{I}/\mathscr{I}^2)\otimes f^*\Omega^q_{X/Z} \to i'^*\Omega^{q+d}_{Y'/Z}$$

represent β_1, β_2 respectively; hence $\beta_1 = \beta_2$. □

Lemma 4. *Let $i: X \to Y$ be an admissible closed embedding. Then i_* satisfies the projection formula, i.e., for all $x \in H^p(X, \Omega^q_{X/Z})$, $y \in H^r(Y, \Omega^s_{Y/Z})$, we have*

$$i_*x \cup y = i_*(x \cup i^*y) \in H^{p+r+d}_X(Y, \Omega^{q+s+d}_{Y/Z}).$$

Here $d = \dim Y - \dim X$, and $\cup$ denotes any one of the (composite) cup products

$$H^p(X,\Omega^q_{X/Z})\otimes H^r(X,\Omega^s_{X/Z}) \to H^{p+r}(X,\Omega^q_{X/Z}\otimes\Omega^s_{X/Z}) \to H^{p+r}(X,\Omega^{q+s}_{X/Z})$$

or

$$H^{p+d}_X(Y,\Omega^{q+d}_{Y/Z})\otimes H^r(Y,\Omega^s_{Y/Z}) \to H^{p+r+d}_X(Y,\Omega^{q+d}_{Y/Z}\otimes\Omega^s_{Y/Z}) \to H^{p+r+d}_X(Y,\Omega^{q+s+d}_{Y/Z})$$

Proof. It will be convenient to consider a cup product

$$\operatorname{Ext}^k_Y(\mathscr{G},\mathscr{H})\otimes H^l(Y,\mathscr{F}) \to \operatorname{Ext}^{k+l}_Y(\mathscr{G},\mathscr{H}\otimes\mathscr{F})$$

where $\mathscr{F}$, $\mathscr{G}$ are quasi-coherent $\mathcal{O}_Y$-modules, and $\mathscr{H}$ is locally free (possibly of infinite rank); this will have obvious functorial properties in the three variables. To construct it, fix $\mathscr{G}$, $\mathscr{H}$ and regard $\{\mathrm{Ext}^{k+l}_Y(\mathscr{G}, \mathscr{H} \otimes \mathscr{F})\}_{l \geqslant 0}$ as a covariant ∂-functor in the variable $\mathscr{F}$ (see [G]). Since $\{H^l(Y, \mathscr{F})\}_{l \geqslant 0}$ is a universal ∂-functor, we are reduced to constructing such a pairing for $l = 0$, with appropriate functorial properties. If $a \in \mathrm{Ext}^k_Y(\mathscr{G}, \mathscr{H})$, $b \in H^0(Y, \mathscr{F})$, then $1 \otimes b : \mathscr{H} \to \mathscr{H} \otimes \mathscr{F}$ induces

$$(1 \otimes b)_* : \mathrm{Ext}^k_Y(\mathscr{G}, \mathscr{H}) \to \mathrm{Ext}^k_Y(\mathscr{G}, \mathscr{H} \otimes \mathscr{F});$$

we set $a \cup b = (1 \otimes b)_* a$. This is the desired pairing when $l = 0$.

The expression $i_* x \cup y$ is the image of

$$x \otimes y \in H^p(X, \Omega^q_{X/\mathbf{Z}}) \otimes H^r(Y, \Omega^s_{Y/\mathbf{Z}})$$

under the following composition of maps:

$$H^p(X, \Omega^q_{X/\mathbf{Z}}) \otimes H^r(Y, \Omega^s_{Y/\mathbf{Z}}) \to \mathrm{Ext}^p_Y(i_* \mathcal{O}_X, i_* \Omega^q_{X/\mathbf{Z}}) \otimes H^r(Y, \Omega^s_{Y/\mathbf{Z}}) \xrightarrow{\cdot \alpha_q(i) \otimes 1}$$
$$\mathrm{Ext}^{p+d}_Y(i_* \mathcal{O}_X, \Omega^{q+d}_{Y/\mathbf{Z}}) \otimes H^r(Y, \Omega^s_{Y/\mathbf{Z}}) \xrightarrow{\psi} \mathrm{Ext}^{p+r+d}_Y(i_* \mathcal{O}_X, \Omega^{q+d}_{Y/\mathbf{Z}} \otimes \Omega^s_{Y/\mathbf{Z}})$$
$$\to H^{p+r+d}_X(Y, \Omega^{q+d}_{Y/\mathbf{Z}} \otimes \Omega^s_{Y/\mathbf{Z}}) \to H^{p+r+d}_X(Y, \Omega^{q+s+d}_{Y/\mathbf{Z}}).$$

Here, ψ is the extended cup product constructed above; we have used the fact that the cup product in local cohomology

$$H^{p+d}_X(Y, \Omega^{q+d}_{Y/\mathbf{Z}}) \otimes H^r(Y, \Omega^s_{Y/\mathbf{Z}}) \xrightarrow{\cup} H^{p+d+r}_X(Y, \Omega^{q+d}_{Y/\mathbf{Z}} \otimes \Omega^s_{Y/\mathbf{Z}})$$

is compatible with ψ (again, this follows from the universality of the ∂-functor $\{H^r(Y, \mathscr{F})\}$). On the other hand, the expression $i_*(x \cup i^* y)$ is the image of $x \otimes y$ under the following composite:

$$H^p(X, \Omega^q_{X/\mathbf{Z}}) \otimes H^r(Y, \Omega^s_{Y/\mathbf{Z}}) \xrightarrow{1 \otimes i^*} H^p(X, \Omega^q_{X/\mathbf{Z}}) \otimes H^r(X, \Omega^s_{X/\mathbf{Z}}) \xrightarrow{\cup} H^{p+r}(X, \Omega^{q+s}_{X/\mathbf{Z}}$$
$$\to \mathrm{Ext}^{p+r}_Y(i_* \mathcal{O}_X, i_* \Omega^{q+s}_{X/\mathbf{Z}}) \xrightarrow{\cdot \alpha_{q+s}(i)} \mathrm{Ext}^{p+r+d}_Y(i_* \mathcal{O}_X, \Omega^{q+s+d}_{Y/\mathbf{Z}})$$
$$\to H^{p+s+d}_X(Y, \Omega^{q+s+d}_{Y/\mathbf{Z}}).$$

We will show that $i_* x \cup y = i_*(x \cup i^* y)$ holds as an equation in

$$\mathrm{Ext}^{p+r+d}_Y(i_* \mathcal{O}_X, \Omega^{q+s+d}_{Y/\mathbf{Z}}).$$

In fact, we find it convenient to prove a stronger statement. Let $\mathscr{G}$ be any quasi coherent $\mathcal{O}_X$-module. Then we have pairings, which reduce to the above two composite pairings for $\mathscr{G} = \mathcal{O}_X$, defined as follows:

$$\mathrm{Ext}^p_X(\mathscr{G}, \Omega^q_{X/\mathbf{Z}}) \otimes H^r(Y, \Omega^s_{Y/\mathbf{Z}}) \to \mathrm{Ext}^p_Y(i_* \mathscr{G}, i_* \Omega^q_{X/\mathbf{Z}}) \otimes H^r(Y, \Omega^s_{Y/\mathbf{Z}})$$
$$\xrightarrow{(\cdot \alpha_q(i)) \otimes 1} \mathrm{Ext}^{p+d}_Y(i_* \mathscr{G}, \Omega^{q+d}_{Y/\mathbf{Z}}) \otimes H^r(Y, \Omega^s_{Y/\mathbf{Z}}) \to \mathrm{Ext}^{p+r+d}_Y(i_* \mathscr{G}, \Omega^{q+s+d}_{Y/\mathbf{Z}}),$$

and similarly

$$\operatorname{Ext}^p_X(\mathscr{G},\Omega^q_{X/Z})\otimes H^r(Y,\Omega^s_{Y/Z})\to\operatorname{Ext}^p_X(\mathscr{G},\Omega^q_{X/Z})\otimes H^r(X,\Omega^s_{X/Z})\overset{\cup}{\to}$$

$$\operatorname{Ext}^{p+r}_X(\mathscr{G},\Omega^{q+s}_{X/Z})\to\operatorname{Ext}^{p+r}_Y(i_*\mathscr{G},i_*\Omega^{q+s}_{X/Z})\xrightarrow{\cdot\alpha_{q+s}(i)}\operatorname{Ext}^{p+r+d}_Y(i_*\mathscr{G},\Omega^{q+s+d}_{Y/Z}).$$

For a fixed $y\in H^r(Y,\Omega^s_{Y/Z})$ we may regard the two resulting maps

$$\operatorname{Ext}^p_X(\mathscr{G},\Omega^q_{X/Z})\to\operatorname{Ext}^{p+r+d}_Y(i_*\mathscr{G},\Omega^{q+s+d}_{Y/Z})$$

as natural transformations of ∂-functors in the variable $\mathscr{G}$, since each arrow in the above compositions may be so regarded.

We use the following terminology: a ∂-functor $T=\{T_n\}_{n\geqslant 0}$ is *weakly universal* if any two natural transformations of ∂-functors $T\to T'$ which agree on T_0 are equal. The ∂-functor $\{\operatorname{Ext}^p_X(\mathscr{G},\Omega^q_{X/Z})\}_{p\geqslant 0}$ is weakly universal, since for any $\mathscr{G}$, and any $a\in\operatorname{Ext}^p_X(\mathscr{G},\Omega^q_{X/Z})$, there is a surjection $\mathscr{G}'\twoheadrightarrow\mathscr{G}$ such that $a\mapsto 0\in\operatorname{Ext}^p_X(\mathscr{G}',\Omega^q_{X/Z})$.

Hence, we see that in order to prove the equality of the two pairings, it suffices to prove it for $p=0$; further, for each $y\in H^r(Y,\Omega^s_{Y/Z})$, it suffices to prove that the two maps

$$\operatorname{Hom}_X(\Omega^q_{X/Z},\Omega^q_{X/Z})\to\operatorname{Ext}^{r+d}_Y(i_*\Omega^q_{X/Z},\Omega^{q+s+d}_{Y/Z})$$

have the same value on the identity endomorphism of $\Omega^q_{X/Z}$ ("Yoneda's lemma").

Thus we are reduced to proving the equality of the following two composites:

$$H^r(Y,\Omega^s_{Y/Z})\xrightarrow{\alpha_q(i)\cup -}\operatorname{Ext}^{r+d}_Y(i_*\Omega^q_{X/Z},\Omega^{q+d}_{Y/Z}\otimes\Omega^s_{Y/Z})\to\operatorname{Ext}^{r+d}_Y(i_*\Omega^q_X,\Omega^{q+d+s}_{Y/Z})$$

and

$$H^r(Y,\Omega^s_{Y/Z})\overset{i^*}{\to}H^r(X,\Omega^s_{X/Z})\overset{\mu}{\to}\operatorname{Ext}^r_X(\Omega^q_{X/Z},\Omega^q_{X/Z}\otimes\Omega^s_{X/Z})$$

$$\to\operatorname{Ext}^r_X(\Omega^q_{X/Z},\Omega^{q+s}_{X/Z})\to\operatorname{Ext}^r_Y(i_*\Omega^q_{X/Z},i_*\Omega^{q+s}_{X/Z})\xrightarrow{\cdot\alpha_{q+s}(i)}\operatorname{Ext}^{r+d}_Y(i_*\Omega^q_{X/Z},\Omega^{q+s+d}_{Y/Z}).$$

Here μ is the (refined) cup product with

$$1\in\operatorname{Ext}^0_X(\Omega^q_{X/Z},\Omega^q_{X/Z})=\operatorname{Hom}_X(\Omega^q_{X/Z},\Omega^q_{X/Z}).$$

We first modify the description of the second map, as follows. If $\mathscr{I}$ is the ideal sheaf of $i(X)\subset Y$, then we have a commutative diagram

$$\begin{tikzcd}
i^*\overset{d}{\wedge}(\mathscr{I}/\mathscr{I}^2)\otimes\Omega^q_{X/Z}\otimes i^*\Omega^s_{Y/Z} \arrow[r, "\varphi_q\otimes 1"] \arrow[d] & i^*\Omega^{q+d}_{Y/Z}\otimes i^*\Omega^s_{Y/Z} \arrow[d] \\
i^*\overset{d}{\wedge}(\mathscr{I}/\mathscr{I}^2)\otimes\Omega^{q+s}_{X/Z} \arrow[r, "\varphi_{q+s}"] & i^*\Omega^{q+s+d}_{Y/Z}
\end{tikzcd}$$

where

$$\varphi_j:i^*\overset{d}{\wedge}(\mathscr{I}/\mathscr{I}^2)\otimes\Omega^j_{X/Z}\to i^*\Omega^{j+d}_{Y/Z}$$

is the natural map induced from the exact sequence

$$0\to i^*\mathscr{I}/\mathscr{I}^2\to i^*\Omega^1_{Y/Z}\to\Omega^1_{X/Z}\to 0.$$

Hence, if $\alpha_q(i)\mapsto\overline{\alpha_q(i)}$ under the map

$$\mathrm{Ext}^d_Y(i_*\Omega^q_{X/Z},\Omega^{q+d}_{Y/Z})\to\mathrm{Ext}^d_Y(i_*\Omega^q_{X/Z}\otimes\Omega^s_{Y/Z},\Omega^{q+d}_{Y/Z}\otimes\Omega^s_{Y/Z}),$$

then the following square (whose horizontal arrow are Yoneda products with $\overline{\alpha_q(i)}$, $\alpha_{q+s}(i)$ respectively) is commutative:

$$\begin{tikzcd}
\mathrm{Ext}^r_Y(i_*\Omega^q_{X/Z},i_*\Omega^q_{X/Z}\otimes\Omega^s_{Y/Z}) \arrow[r, "\cdot\overline{\alpha_q(i)}"] \arrow[d] & \mathrm{Ext}^{r+d}_Y(i_*\Omega^q_{X/Z},\Omega^{q+d}_{Y/Z}\otimes\Omega^s_{Y/Z}) \arrow[d] \\
\mathrm{Ext}^r_Y(i_*\Omega^q_{X/Z},i_*\Omega^{q+s}_{X/Z}) \arrow[r, "\cdot\alpha_{q+s}(i)"] & \mathrm{Ext}^{r+d}_Y(i_*\Omega^q_{X/Z},\Omega^{q+d+s}_{Y/Z})
\end{tikzcd}$$

Thus, we are reduced to proving the commutativity of the following diagram (for each $r,s\geqslant 0$):

$$\begin{tikzcd}
H^r(Y,\Omega^s_{Y/Z}) \arrow[rr, "\alpha_q(i)\cup -"] \arrow[d] & & \mathrm{Ext}^{r+d}_Y(i_*\Omega^q_{X/Z},\Omega^{q+d}_{Y/Z}\otimes\Omega^s_{Y/Z}) \\
H^r(X,i^*\Omega^s_{Y/Z}) \arrow[dr] & & \mathrm{Ext}^r_Y(i_*\Omega^q_{X/Z},i_*\Omega^q_{X/Z}\otimes\Omega^s_{Y/Z}) \arrow[u, "\cdot\overline{\alpha_q(i)}"'] \\
& \mathrm{Ext}^r_X(\Omega^q_{X/Z},\Omega^q_{X/Z}\otimes i^*\Omega^s_{Y/Z}) \arrow[ur] &
\end{tikzcd}$$

Again, we find it convenient to prove a more general statement. Let $\mathscr{E}$, $\mathscr{F}$ be locally free $\mathcal{O}_Y$-modules, and $\mathscr{G}$ be a locally free $\mathcal{O}_X$-module, and let $\alpha\in\mathrm{Ext}^d_Y(i_*\mathscr{G},\mathscr{F})$. Let $\alpha\mapsto\bar{\alpha}$ under the natural map

$$\mathrm{Ext}^d_Y(i_*\mathscr{G},\mathscr{F})\to\mathrm{Ext}^d_Y(i_*\mathscr{G}\otimes\mathscr{E},\mathscr{F}\otimes\mathscr{E})$$

(which we may view as obtained by applying the exact functor $-\otimes\mathscr{E}$ to d-extensions). Then we claim the following diagram commutes:

$$\begin{tikzcd}
H^r(Y,\mathscr{E}) \arrow[rr, "\alpha\cup -"] \arrow[d] & & \mathrm{Ext}^{r+d}_Y(i_*\mathscr{G},\mathscr{F}\otimes\mathscr{E}) \\
H^r(X,i^*\mathscr{E}) \arrow[dr] & \odot & \mathrm{Ext}^r_Y(i_*\mathscr{G},i_*\mathscr{G}\otimes\mathscr{E}) \arrow[u, "\cdot\bar{\alpha}"'] \\
& \mathrm{Ext}^r_X(\mathscr{G},\mathscr{G}\otimes i^*\mathscr{E}) \arrow[ur] &
\end{tikzcd}$$

If we fix $\mathscr{G}$, $\mathscr{F}$ then all of the groups in the above diagram yield ∂-functors in the variable $\mathscr{E}$ (in the sense that given a short exact sequence of $\mathscr{E}$'s we have appropriate long exact sequences), and the arrows of the diagram may be regarded as morphisms between ∂-functors. We claim that $\{H^r(Y,\mathscr{E})\}_{r\geqslant 0}$ is a weakly universal ∂-functor, so that it suffices to check the commutativity of the diagram for $r=0$. The weak universality follows from:

Sublemma 1. *Let Y be a scheme on which every quasi-coherent $\mathcal{O}_Y$-module is a quotient of a locally free $\mathcal{O}_Y$ module (e.g. a regular scheme). Then for any, locally free $\mathcal{O}_Y$ module $\mathscr{E}$ and any $\beta\in H^r(Y,\mathscr{E})$, where $r>0$, there is a short exact sequence of locally free $\mathcal{O}_Y$-modules*

$$0\to\mathscr{E}\to\mathscr{E}'\to\mathscr{E}''\to 0$$

such that $\beta \mapsto 0$ *under* $H^r(Y,\mathscr{E}) \to H^r(Y,\mathscr{E}')$.

Proof. The class $\beta \in H^r(Y,\mathscr{E}) = \operatorname{Ext}^r_Y(\mathcal{O}_Y,\mathscr{E})$ is represented by an r-extension of quasi-coherent $\mathcal{O}_Y$-modules

$$0 \to \mathscr{E} \to \mathscr{F}_r \to \mathscr{F}_{r-1} \to \cdots \to \mathscr{F}_1 \to \mathcal{O}_Y \to 0.$$

We claim that there is an equivalent r-extension where the $\mathscr{F}_i$ are locally free $\mathcal{O}_Y$-modules. This is clear for $r = 1$, so let $r > 1$. By a standard argument, we see that there is a resolution of $\mathcal{O}_Y$ by locally free $\mathcal{O}_Y$-modules

$$\mathscr{F}'_{r-1} \xrightarrow{f'} \mathscr{F}'_{r-2} \to \cdots \to \mathscr{F}'_1 \to \mathcal{O}_Y \to 0$$

together with a diagram with surjective vertical maps

$$\begin{tikzcd}
 & & \mathscr{F}'_{r-1} \arrow[r, "f'"] \arrow[d] & \mathscr{F}'_{r-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathscr{F}'_1 \arrow[r] \arrow[d] & \mathcal{O}_Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{E} \arrow[r] & \mathscr{F}_r \arrow[r] & \mathscr{F}_{r-1} \arrow[r, "f"] & \mathscr{F}_{r-2} \arrow[r] & \cdots \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathcal{O}_Y \arrow[r] & 0,
\end{tikzcd}$$

such that if $\mathscr{E}'' = \ker f'$, then $\mathscr{E}''$ is locally free, and the natural map $h:\mathscr{E}'' \to \ker f$ is surjective. Let $\mathscr{E}'$ be defined by the short exact sequence obtained from

$$0 \to \mathscr{E} \to \mathscr{F}_{r-1} \to \ker f \to 0$$

by pullback along h; then we have a short exact sequence

$$0 \to \mathscr{E} \to \mathscr{E}' \to \mathscr{E}'' \to 0$$

(so that $\mathscr{E}'$ is locally free), and a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E} \arrow[r] \arrow[d, equal] & \mathscr{E}' \arrow[r] \arrow[d] & \mathscr{F}'_{r-1} \arrow[r, "f'"] \arrow[d] & \mathscr{F}'_{r-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathscr{F}'_1 \arrow[r] \arrow[d] & \mathcal{O}_Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{E} \arrow[r] & \mathscr{F}_r \arrow[r] & \mathscr{F}_{r-1} \arrow[r, "f"] & \mathscr{F}_{r-2} \arrow[r] & \cdots \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathcal{O}_Y \arrow[r] & 0
\end{tikzcd}$$

It is immediate that the class $\beta \in \operatorname{Ext}^n_Y(\mathcal{O}_Y,\mathscr{E})$ maps to 0 in $\operatorname{Ext}^n_Y(\mathcal{O}_Y,\mathscr{E}')$, since the pushout of

$$0 \to \mathscr{E} \xrightarrow{g} \mathscr{E}' \to \mathscr{E}'' \to 0$$

along $g:\mathscr{E} \to \mathscr{E}'$ is split. □

We resume the proof of the projection formula (lemma 4). We have only to verify the commutativity of the diagram ⊙ for $r = 0$. Let $\beta \in H^0(Y,\mathscr{E})$, $\beta:\mathcal{O}_Y \to \mathscr{E}$. Then $\alpha \cup \beta$ is the image of α under

$$(1 \otimes \beta)_*:\operatorname{Ext}^d_Y(i_*\mathscr{G},\mathscr{F}) \to \operatorname{Ext}^d_Y(i_*\mathscr{G},\mathscr{F} \otimes \mathscr{E});$$

on the other hand, the composite

$$H^0(Y,\mathscr{E})\to H^0(X,i^*\mathscr{E})\to \mathrm{Hom}_X(\mathscr{G},\mathscr{G}\otimes i^*\mathscr{E})\to \mathrm{Hom}_Y(i_*\mathscr{G},i_*\mathscr{G}\otimes\mathscr{E})$$

$$\xrightarrow{\cdot\bar{\alpha}} \mathrm{Ext}^d_Y(i_*\mathscr{G},\mathscr{F}\otimes\mathscr{E})$$

maps β to $\tilde{\alpha}$, where $\bar{\alpha}\mapsto\tilde{\alpha}$ under

$$\mathrm{Ext}^d_Y(i_*\mathscr{G}\otimes\mathscr{E},\mathscr{F}\otimes\mathscr{E})\xrightarrow{(1\otimes\beta)^*}\mathrm{Ext}^d_Y(i_*\mathscr{G},\mathscr{F}\otimes\mathscr{E})$$

(here $1\otimes\beta:\mathscr{F}\to\mathscr{F}\otimes\mathscr{E}$). We must show $\alpha\cup\beta=\tilde{\alpha}$.

We give a proof in the case $d>0$ (a minor modification, left to the reader, works for $d=0$). Let $\alpha\in\mathrm{Ext}^d_Y(i_*\mathscr{G},\mathscr{F})$ correspond to the d-extension

$$0\to\mathscr{F}\to\mathscr{F}_d\to\cdots\to\mathscr{F}_1\to i_*\mathscr{G}\to 0.$$

Then $\alpha\cup\beta$ is the d-extension obtained by pushout along

$$1\otimes\beta:\mathscr{F}\to\mathscr{F}\otimes\mathscr{E},$$

so that we have a diagram with exact rows (whose bottom row represents $\alpha\cup\beta$)

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d, "1\otimes\beta"'] & \mathscr{F}_d \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathscr{F}_1 \arrow[r] \arrow[d, equal] & i_*\mathscr{G} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{F}\otimes\mathscr{E} \arrow[r] & \mathscr{H} \arrow[r] & \cdots \arrow[r] & \mathscr{F}_1 \arrow[r] & i_*\mathscr{G} \arrow[r] & 0
\end{tikzcd}$$

where all the vertical arrows, except the first two on the left, are the identity. Next, $\bar{\alpha}$ is represented by

$$0\to\mathscr{F}\otimes\mathscr{E}\to\mathscr{F}_d\otimes\mathscr{E}\to\cdots\to\mathscr{F}_1\otimes\mathscr{E}\to i_*\mathscr{G}\otimes\mathscr{E}\to 0,$$

so that $\tilde{\alpha}$ is represented by the pullback along $1\otimes\beta:i_*\mathscr{G}\to i_*\mathscr{G}\otimes\mathscr{E}$; thus we have a diagram with exact rows (whose top row represents $\tilde{\alpha}$)

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}\otimes\mathscr{E} \arrow[r] \arrow[d, equal] & \mathscr{F}_d\otimes\mathscr{E} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \mathscr{H}' \arrow[r] \arrow[d] & i_*\mathscr{G} \arrow[r] \arrow[d, "1\otimes\beta"] & 0 \\
0 \arrow[r] & \mathscr{F}\otimes\mathscr{E} \arrow[r] & \mathscr{F}_d\otimes\mathscr{E} \arrow[r] & \cdots \arrow[r] & \mathscr{F}_1\otimes\mathscr{E} \arrow[r] & i_*\mathscr{G}\otimes\mathscr{E} \arrow[r] & 0
\end{tikzcd}$$

where all the vertical arrows, except the last two on the right, are the identity. We have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}_d \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathscr{F}_1 \arrow[r] \arrow[d] & i_*\mathscr{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}\otimes\mathscr{E} \arrow[r] & \mathscr{F}_d\otimes\mathscr{E} \arrow[r] & \cdots \arrow[r] & \mathscr{F}_1\otimes\mathscr{E} \arrow[r] & i_*\mathscr{G}\otimes\mathscr{E} \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows all have the form $1\otimes\beta$. From the universal properties of pushouts and pullbacks, we see that there is an induced diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}\otimes\mathscr{E} \arrow[r] \arrow[d, equal] & \mathscr{H} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathscr{F}_1 \arrow[r] \arrow[d] & i_*\mathscr{G} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{F}\otimes\mathscr{E} \arrow[r] & \mathscr{F}_d\otimes\mathscr{E} \arrow[r] & \cdots \arrow[r] & \mathscr{H}' \arrow[r] & i_*\mathscr{G} \arrow[r] & 0
\end{tikzcd}$$

This shows that the d-extensions representing $\alpha\cup\beta$ and $\tilde{\alpha}$ are equivalent. □

Let Y be a Noetherian integral separated regular scheme such that $\Omega^1_{Y/\mathbf{Z}}$ is locally free, and let $\mathscr{E}$ be a locally free $\mathcal{O}_Y$-module of rank $m+1$. Then the structure morphism $\pi:\mathbf{P}_Y(\mathscr{E})\to Y$ is admissible. Let $Z\subset Y$ be a closed subscheme, and let $W=\pi^{-1}(Z)$. We define Gysin maps (for $p,q\geqslant m$)

$$\pi_*:H^p_W(\mathbf{P}_Y(\mathscr{E}),\Omega^q_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}})\to H^{p-m}_Z(Y,\Omega^{q-m}_{Y/\mathbf{Z}})$$

as follows: there is a (Leray) spectral sequence

$$E_2^{r,s}=H^r_Z(Y,R^s\pi_*\Omega^q_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}})\Rightarrow H^{r+s}_W(\mathbf{P}_Y(\mathscr{E}),\Omega^q_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}}),$$

where $R^s\pi_*\Omega^q_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}}=0$ for $s>m$; hence there is an edge homomorphism

$$H^p_W(\mathbf{P}_Y(\mathscr{E}),\Omega^q_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}})\to H^{p-m}_Z(Y,R^m\pi_*\Omega^q_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}}).$$

The exact sequence

$$0\to\pi^*\Omega^1_{Y/\mathbf{Z}}\to\Omega^1_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}}\to\Omega^1_{\mathbf{P}_Y(\mathscr{E})/Y}\to 0$$

yields a map

$$\Omega^q_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}}\to\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y}\otimes\pi^*\Omega^{q-m}_{Y/\mathbf{Z}},$$

and hence a composite map

$$\begin{aligned}R^m\pi_*\Omega^q_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}}&\to R^m\pi_*\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y}\otimes\Omega^{q-m}_{Y/\mathbf{Z}}\quad\text{(projection formula)}\\&\cong\Omega^{q-m}_{Y/\mathbf{Z}}\end{aligned}$$

where the last isomorphism is from the trace isomorphism

$$\mathrm{tr}:R^m\pi_*\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y}\cong\mathcal{O}_Y$$

of Grothendieck duality for the morphism π (for example, see [H], III, Exercise (8.4)). Our Gysin map is defined to be the composite

$$H^p_W(\mathbf{P}_Y(\mathscr{E}),\Omega^q_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}})\xrightarrow{\text{edge}}H^{p-m}_Z(Y,R^m\pi_*\Omega^q_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}})\to H^{p-m}_Z(Y,\Omega^{q-m}_{Y/\mathbf{Z}})$$

(for p, $q\geqslant m$; otherwise define $\pi_*=0$).

Lemma 5. 1. *If $Z\subset Z'\subset Y$, and $W'=\pi^{-1}(Z')$, we have a commutative diagram*

$$\begin{tikzcd}H^p_W(\mathbf{P}_Y(\mathscr{E}),\Omega^q_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}}) \arrow[r, "\pi_*"] \arrow[d] & H^{p-m}_Z(Y,\Omega^{q-m}_{Y/\mathbf{Z}}) \arrow[d] \\ H^p_{W'}(\mathbf{P}_Y(\mathscr{E}),\Omega^q_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}}) \arrow[r, "\pi_*"] & H^{p-m}_{Z'}(Y,\Omega^{q-m}_{Y/\mathbf{Z}})\end{tikzcd}$$

where the vertical maps are the natural maps ("increase supports").

(This includes the case $Z'=Y$, $W'=\mathbf{P}_Y(\mathscr{E})$.)

2. *If $f:Y'\to Y$ is a morphism, so that we have a fibre square*

$$\begin{tikzcd}\mathbf{P}_{Y'}(f^*\mathscr{E}) \arrow[r, "f'"] \arrow[d, "\pi'"'] & \mathbf{P}_Y(\mathscr{E}) \arrow[d, "\pi"] \\ Y' \arrow[r, "f"] & Y\end{tikzcd}$$

such that π, π' *are admissible, and if* $Z \subset Y$ *is a closed subscheme,* $Z' = f^{-1}(Z)$ $W = \pi^{-1}(Z)$, $W' = pi'^{-1}(Z') = f'^{-1}(W)$, *then*

$$f^* \circ \pi_* = \pi'_* \circ f'^* : H^p_W(\mathbf{P}_Y(\mathscr{E}), \Omega^q_{\mathbf{P}_{Y(\mathscr{E})/\mathbf{Z}}}) \to H^{p-m}_{Z'}(Y', \Omega^q_{Y'/\mathbf{Z}}).$$

The proof of this lemma is left to the reader (the second statement follows from th compatibility of the trace isomorphism with arbitrary base change).

Lemma 6. *Let* $\mathscr{E}$ *be a locally free* $\mathscr{O}_Y$*-module of rank* $m+1$, *such that* $\pi : \mathbf{P}_Y(\mathscr{E}) \to Y$ *i admissible. Let* $Z \subset Y$ *be a closed subscheme,* $W = \pi^{-1}(Z)$. *Then* π_*, *defined above satisfies the projection formula i.e., for any* $x \in H^p_W(\mathbf{P}_Y(\mathscr{E}), \Omega^q_{\mathbf{P}_{Y(\mathscr{E})/\mathbf{Z}}})$ *and* $y \in H^r(Y, \Omega^s_{Y/\mathbf{Z}}$ *we have*

$$\pi_*(x \cup \pi^* y) = \pi_* x \cup y \in H^{p+r-m}_Z(Y, \Omega^{q+s-m}_{Y/\mathbf{Z}}).$$

Proof. Fix $y \in H^r(Y, \Omega^s_{Y/\mathbf{Z}})$. Clearly the map

$$H^p_W(\mathbf{P}_Y(\mathscr{E}), \Omega^q_{\mathbf{P}_{Y(\mathscr{E})/\mathbf{Z}}}) \to H^{p+r-m}_Z(Y, \Omega^{q+s-m}_{Y/\mathbf{Z}}),$$
$$x \mapsto \pi_* x \cup y,$$

factors through the natural map

$$\psi : H^p_W(\mathbf{P}_Y(\mathscr{E}), \Omega^q_{\mathbf{P}_{Y(\mathscr{E})/\mathbf{Z}}}) \to H^p_W(\mathbf{P}_Y(\mathscr{E}), \Omega^m_{\mathbf{P}_{Y(\mathscr{E})/Y}} \otimes \pi^* \Omega^{q-m}_{Y/\mathbf{Z}}),$$

from the definition of π_*. We claim that $x \mapsto \pi_*(x \cup \pi^* y)$ also factors through ψ (i particular, it is zero if $q < m$). This follows from the commutativity of the square

$$\begin{tikzcd}
H^p_W(\mathbf{P}_Y(\mathscr{E}), \Omega^q_{\mathbf{P}_{Y(\mathscr{E})/\mathbf{Z}}}) \arrow[r, "\psi"] \arrow[d, "- \cup \pi^* y"'] & H^p_W(\mathbf{P}_Y(\mathscr{E}), \Omega^m_{\mathbf{P}_{Y(\mathscr{E})/Y}} \otimes \pi^* \Omega^{q-m}_{Y/\mathbf{Z}}) \arrow[d, "- \cup \pi^* y"] \\
H^{p+r}_W(\mathbf{P}_Y(\mathscr{E}), \Omega^{q+s}_{\mathbf{P}_{Y(\mathscr{E})/\mathbf{Z}}}) \arrow[r] & H^{p+r}_W(\mathbf{P}_Y(\mathscr{E}), \Omega^m_{\mathbf{P}_{Y(\mathscr{E})/Y}} \otimes \pi^* \Omega^{q+s-m}_{Y/\mathbf{Z}})
\end{tikzcd}$$

(where for the right hand vertical arrow, we regard $\pi^* y$ as an element $H^r_W(\mathbf{P}_Y(\mathscr{E}), \pi^* \Omega^s_{Y/\mathbf{Z}})$). Then it suffices to prove the projection formula $\pi_*(x \cup \pi^* y)$ $\pi_* x \cup y$ for $x \in H^p_W(\mathbf{P}_Y(\mathscr{E}), \Omega^m_{\mathbf{P}_{Y(\mathscr{E})/Y}} \otimes \pi^* \Omega^{q-m}_{Y/\mathbf{Z}})$, where π_* is thought of as the ma (defined for each $p, q \geqslant m$)

$$\pi_* : H^p_W(\mathbf{P}_Y(\mathscr{E}), \Omega^m_{\mathbf{P}_{Y(\mathscr{E})/Y}} \otimes \pi^* \Omega^{q-m}_{Y/\mathbf{Z}}) \to H^{p-m}_Z(Y, \Omega^{q-m}_{Y/\mathbf{Z}})$$

induced by the trace isomorphism $R^m \pi_* \Omega^m_{\mathbf{P}_{Y(\mathscr{E})/Y}} \cong \mathscr{O}_Y$, and is defined to be zero one of p, q is $< m$. But

$$R^t \pi_*(\Omega^m_{\mathbf{P}_{Y(\mathscr{E})/Y}} \otimes \pi^* \Omega^{q-m}_{Y/\mathbf{Z}}) \cong R^t \pi_* \Omega^m_{\mathbf{P}_{Y(\mathscr{E})/Y}} \otimes \Omega^{q-m}_{Y/\mathbf{Z}} = 0, \forall t \neq m;$$

hence π_* is an *isomorphism* for each $p, q \geqslant 0$.

Let $x_0 \in H^m(\mathbf{P}_Y(\mathscr{E}), \Omega^m_{\mathbf{P}_{Y(\mathscr{E})/Y}})$ such that $x_0 \mapsto 1$ under the trace isomorphism

$$\pi_* : H^m(\mathbf{P}_Y(\mathscr{E}), \Omega^m_{\mathbf{P}_{Y(\mathscr{E})/Y}}) \to H^0(Y, \mathscr{O}_Y).$$

We claim that the inverse isomorphism to

$$\pi_* : H^p_W(\mathbf{P}_Y(\mathscr{E}), \Omega^m_{\mathbf{P}_{Y(\mathscr{E})/Y}} \otimes \pi^* \Omega^{q-m}_{Y/\mathbf{Z}}) \to H^{p-m}_Z(Y, \Omega^{q-m}_{Y/\mathbf{Z}})$$

is the map

$$H_Z^{p-m}(Y,\Omega^{q-m}_{Y/Z}) \to H^p_W(\mathbf{P}_Y(\mathcal{E}),\Omega^m_{\mathbf{P}_Y(\mathcal{E})/Y}\otimes\pi^*\Omega^{q-m}_{Y/Z}),$$
$$x'\mapsto x_0\cup\pi^*x';$$

thus $x = x_0\cup\pi_*\pi^*x$.

The claim follows from the following standard result.

Sublemma 2. For any quasi-coherent $\mathcal{O}_Y$-module $\mathcal{F}$, the composite

$$H^r_Z(Y,\mathcal{F}) \xrightarrow{\pi^*} H^r_W(\mathbf{P}_Y(\mathcal{E}),\pi^*\mathcal{F}) \xrightarrow{x_0\cup -} H^{r+m}_W(\mathbf{P}_Y(\mathcal{E}),\Omega^m_{\mathbf{P}_Y(\mathcal{E})/Y}\otimes\pi^*\mathcal{F})$$
$$\to H^r_Z(Y,R^m\pi_*(\Omega^m_{\mathbf{P}_Y(\mathcal{E})/Y}\otimes\pi^*\mathcal{F}))\cong H^r_Z(Y,\mathcal{F})$$

(where the last isomorphism is induced by the trace) is the identity.

Proof. We may view each of the maps in the above composite as a natural transformation of ∂-functors in the variable $\mathcal{F}$. Since $\{H^r_Z(Y,\mathcal{F})\}_{r\geqslant 0}$ is universal, it suffices to prove the sublemma for $r=0$, which is easy—it is the statement that the composite

$$\mathcal{F}\mapsto R^m\pi_*(\Omega^m_{\mathbf{P}_Y(\mathcal{E})/Y}\otimes\pi^*\mathcal{F})\cong(R^m\pi_*\Omega^m_{\mathbf{P}_Y(\mathcal{E})/Y})\otimes\mathcal{F}\cong\mathcal{F}$$

is the identity functor. ☐

Now the proof of the lemma (the projection formula) is easily completed. We have $x = x_0\cup\pi^*\pi_*x$, so that

$$x\cup\pi^*y = x_0\cup\pi^*(\pi_*x\cup y),$$

and hence

$$\pi_*(x\cup\pi^*y)=\pi_*(x_0\cup\pi^*(\pi_*x\cup y))=\pi_*x\cup y.$$

☐

Lemma 7. Let $i\colon X\hookrightarrow Y$ be an admissible closed embedding, $\mathcal{E}$ a locally free $\mathcal{O}_Y$-module of rank $(m+1)$, and $\mathcal{E}'=i^*\mathcal{E}$. Let $i'\colon\mathbf{P}_X(\mathcal{E}')\to\mathbf{P}_Y(\mathcal{E})$ be the natural map, so that we have a fibre product diagram

$$\begin{tikzcd}
\mathbf{P}_X(\mathcal{E}') \arrow[r, "i'"] \arrow[d, "\pi'"'] & \mathbf{P}_Y(\mathcal{E}) \arrow[d, "\pi"] \\
X \arrow[r, "i"] & Y
\end{tikzcd}$$

Let $d=\dim Y-\dim X=\dim\mathbf{P}_Y(\mathcal{E})-\dim\mathbf{P}_X(\mathcal{E}')$, and set $i'(\mathbf{P}_X(\mathcal{E}'))=W$. Then the following diagram commutes (for each $p,q\geqslant m$):

$$\begin{tikzcd}
H^p(\mathbf{P}_X(\mathcal{E}'),\Omega^q_{\mathbf{P}_X(\mathcal{E}')/Z}) \arrow[r, "i'_*"] \arrow[d, "\pi'_*"'] & H^{p+d}_W(\mathbf{P}_Y(\mathcal{E}),\Omega^{p+d}_{\mathbf{P}_Y(\mathcal{E})/Z}) \arrow[d, "\pi_*"] \\
H^{p-m}(X,\Omega^{q-m}_{X/Z}) \arrow[r, "i_*"] & H^{p+d-m}_X(Y,\Omega^{q+d-m}_{Y/Z})
\end{tikzcd}$$

Proof. Clearly $i_* \circ \pi'_*$ factors through the natural map

$$\psi : H^p(\mathbf{P}_X(\mathscr{E}'), \Omega^q_{\mathbf{P}_X(\mathscr{E}')/\mathbf{Z}}) \to H^p(\mathbf{P}_X(\mathscr{E}'), i^*\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes \pi'^*\Omega^{q-m}_{X/\mathbf{Z}})$$

(where we have used the natural identification $i^*\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \cong \Omega^m_{\mathbf{P}_X(\mathscr{E}')/X}$). We claim tha the other composite $\pi_* \circ i'_*$ also factors through ψ.

Let $\alpha_{q-m}(i) \mapsto \bar{\alpha}$ under the composite map

$$\operatorname{Ext}^d_Y(i_*\Omega^{q-m}_{X/\mathbf{Z}}, \Omega^{q-m+d}_{Y/\mathbf{Z}}) \xrightarrow{\pi^*} \operatorname{Ext}^d_{\mathbf{P}_Y(\mathscr{E})}(i'_*\pi'^*\Omega^{q-m}_{X/\mathbf{Z}}, \pi^*\Omega^{q-m+d}_{Y/\mathbf{Z}})$$
$$\to \operatorname{Ext}^d_{\mathbf{P}_Y(\mathscr{E})}(\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes i'_*\pi'^*\Omega^{q-m}_{X/\mathbf{Z}}, \Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes \pi^*\Omega^{q+d-m}_{Y/\mathbf{Z}}).$$

If $\mathscr{I}$ is the ideal sheaf of $i(X) \subset Y$, then $\pi^*\mathscr{I}$ is the ideal sheaf of $W \subset \mathbf{P}_Y(\mathscr{E})$. Each the above Ext groups is identified with an appropriate Hom group, and the operatio π^* and $\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes$ – are compatible with these identifications. Hence, if

$$\varphi_r(i) : i^* \overset{d}{\wedge} (\mathscr{I}/\mathscr{I}^2) \otimes \Omega^r_{X/\mathbf{Z}} \to i^*\Omega^{r+d}_{Y/\mathbf{Z}}$$

is the natural map, then under the identification

$$\operatorname{Ext}^d_{\mathbf{P}_Y(\mathscr{E})}(\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes i'_*\pi'^*\Omega^{q-m}_{X/\mathbf{Z}}, \Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes \pi^*\Omega^{q+d-m}_{Y/\mathbf{Z}}) \cong$$

$$\operatorname{Hom}_{\mathbf{P}_X(\mathscr{E}')}(i'^*\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes i'^*\pi^* \overset{d}{\wedge} (\mathscr{I}/\mathscr{I}^2) \otimes \pi'^*\Omega^{q-m}_{X/\mathbf{Z}}, i'^*\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes i'^*\pi^*\Omega^{q+d-m}_{Y/\mathbf{Z}}),$$

we see that $\bar{\alpha}$ corresponds to the map $1 \otimes \pi^*\varphi_{q-m}(i)$. Thus, under the identificatio (consistent with our earlier such identifications)

$$\operatorname{Ext}^d_{\mathbf{P}_Y(\mathscr{E})}(\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes i'_*\pi'^*\Omega^{q-m}_{X/\mathbf{Z}}, \Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes \pi^*\Omega^{q+d-m}_{Y/\mathbf{Z}}) \cong$$
$$\operatorname{Hom}_{\mathbf{P}_X(\mathscr{E}')}(i'^*\pi^*(\mathscr{I}/\mathscr{I}^2) \otimes i'^*\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes \pi'^*\Omega^{q-m}_{X/\mathbf{Z}}, i'^*(\Omega^m_{\mathbf{P}_Y(\mathscr{E})} \otimes \pi^*\Omega^{q+d-m}_{Y/\mathbf{Z}}))$$

we see that $\bar{\alpha}$ corresponds to

$$\bar{\varphi} : i'^*\pi^*(\mathscr{I}/\mathscr{I}^2) \otimes i'^*\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes \pi'^*\Omega^{q-m}_{X/\mathbf{Z}} \to i'^*(\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes \pi^*\Omega^{q+d-m}_{Y/\mathbf{Z}}),$$
$$\bar{\varphi}(x_1 \otimes x_2 \otimes x_3) = (1 \otimes \pi^*\varphi_{q-m}(i))(x_2 \otimes x_1 \otimes x_3).$$

Now $\alpha_q(i')$ corresponds to

$$\varphi_q(i') : i'^*\pi^* \overset{d}{\wedge} (\mathscr{I}/\mathscr{I}^2) \otimes \Omega^q_{W/\mathbf{Z}} \to i'^*\Omega^{q+d}_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}},$$

and we have a diagram

$$\begin{tikzcd}
i'^*\pi^* \overset{d}{\wedge} (\mathscr{I}/\mathscr{I}^2) \otimes \Omega^q_{W/\mathbf{Z}} \arrow[r] \arrow[d, "\varphi_q(i')"'] & i'^*\pi^*(\mathscr{I}/\mathscr{I}^2) \otimes i'^*\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes \pi'^*\Omega^{q-m}_{X/\mathbf{Z}} \arrow[d, "\bar{\varphi}"] \\
i'^*\Omega^{q+d}_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}} \arrow[r] & i'^*(\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes \pi^*\Omega^{q+d-m}_{Y/\mathbf{Z}})
\end{tikzcd}$$

which commutes upto a sign $(-1)^{md}$. Hence the diagram below commutes upto sign:

$$\begin{tikzcd}
\operatorname{Ext}^p_{\mathbf{P}_Y(\mathscr{E})}(i'_*\mathscr{O}_W, i'_*\Omega^q_{W/\mathbf{Z}}) \arrow[r] \arrow[d, "\cdot\alpha_q(i')"'] & \operatorname{Ext}^p_{\mathbf{P}_Y(\mathscr{E})}(i'_*\mathscr{O}_W, \Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes i'_*\pi'^*\Omega^{q-m}_{X/\mathbf{Z}}) \arrow[d, "\cdot\bar{\alpha}"] \\
\operatorname{Ext}^{p+d}_{\mathbf{P}_Y(\mathscr{E})}(i'_*\mathscr{O}_W, \Omega^{q+d}_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}}) \arrow[r] & \operatorname{Ext}^{p+d}_{\mathbf{P}_Y(\mathscr{E})}(i'_*\mathscr{O}_W, \Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes \pi^*\Omega^{q+d-m}_{Y/\mathbf{Z}}).
\end{tikzcd}$$

Now it is evident that $\pi_* \circ i'_*$ also factors through ψ.

Hence, to prove the lemma, it suffices to prove it for a set of elements $x'\in H^p(\mathbf{P}_X(\mathscr{E}'),\Omega^q_{\mathbf{P}_X(\mathscr{E}')/\mathbf{Z}})$ whose images under ψ generate im ψ.

Let $y_0\in H^1(\mathbf{P}_Y(\mathscr{E}),\Omega^1_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}})$ be the image of the class of $\mathscr{O}_{\mathbf{P}_Y(\mathscr{E})}(1)$ under the map

$$\mathrm{dlog}\colon H^1(\mathbf{P}_Y(\mathscr{E}),\mathscr{O}^*_{\mathbf{P}_Y(\mathscr{E})})\to H^1(\mathbf{P}_Y(\mathscr{E}),\Omega^1_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}}).$$

Then the image of y_0 in $H^1(\mathbf{P}_Y(\mathscr{E}),\Omega^1_{\mathbf{P}_Y(\mathscr{E})/Y})$ is the "usual" (relative) first Chern class of $\mathscr{O}_{\mathbf{P}_Y(\mathscr{E})}(1)$, so that

$$y=y_0^m\in H^m(\mathbf{P}_Y(\mathscr{E}),\Omega^m_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}})$$

maps to $1\in H^0(Y,\mathscr{O}_Y)$ under the map

$$\pi_*\colon H^m(\mathbf{P}_Y(\mathscr{E}),\Omega^m_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}})\to H^m(\mathbf{P}_Y(\mathscr{E}),\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y})\cong H^0(Y,\mathscr{O}_Y).$$

Further, since $i'^*\mathscr{O}_{\mathbf{P}_Y(\mathscr{E})}(1)=\mathscr{O}_{\mathbf{P}_X(\mathscr{E}')}(1)$, a similar calculation yields $\pi'_*i'^*y=1\in H^0(X,\mathscr{O}_X)$.

By the sublemma 2, we see that the elements $i'^*y\cup\pi'^*x'$, with $x'\in H^{p-m}(X,\Omega^{q-m}_{X/\mathbf{Z}})$, have the property that their images under ψ generate

$$H^p(\mathbf{P}_X(\mathscr{E}'),i'^*\Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y}\otimes\pi'^*\Omega^{q-m}_{X/\mathbf{Z}}).$$

Hence it suffices to prove the lemma for elements of this form. From lemmas 3 and 4, we compute that

$$\begin{aligned}\pi_*\circ i'_*(i'^*y\cup\pi'^*x')&=(-1)^{(p+q)m}\pi_*\circ i'_*(\pi'^*x'\cup i'^*y)\\&=(-1)^{(p+q)m}\pi_*(i'_*\circ\pi'^*x'\cup y)\\&=(-1)^{(p+q)m}\pi_*(\pi^*\circ i_*x'\cup y)\\&=\pi_*(y\cup\pi^*\circ i_*x')\\&=i_*x',\end{aligned}$$

and similarly

$$\begin{aligned}i_*\circ\pi'_*(i'^*y\cup\pi'^*x')&=(i_*(\pi'_*\circ i'^*y\cup x')\\&=i_*(1\cup x')\\&=i_*x'\end{aligned}$$

(we have used the commutation rule $x\cup y=(-1)^{pr+qs}y\cup x$ for $x\in H^p(\Omega^q)$, $y\in H^r(\Omega^s)$).

□

Lemma 8. *Let X be a Noetherian integral regular scheme, $\mathscr{E}$ a locally free $\mathscr{O}_X$-module of rank $m+1$, and $\pi\colon\mathbf{P}_X(\mathscr{E})\to X$ the natural map. Suppose that π has a section $i\colon X\to\mathbf{P}_X(\mathscr{E})$. Then for any $p,q\geqslant 0$,*

$$\pi_*\circ\varphi\circ i_*\colon H^p(X,\Omega^q_{X/\mathbf{Z}})\to H^p(X,\Omega^q_{X/\mathbf{Z}})$$

is the identity map, where

$$\varphi\colon H^{p+m}_{i(X)}(\mathbf{P}_X(\mathscr{E}),\Omega^{q+m}_{\mathbf{P}_X(\mathscr{E})/\mathbf{Z}})\to H^{p+m}(\mathbf{P}_X(\mathscr{E}),\Omega^{q+m}_{\mathbf{P}_X(\mathscr{E})/\mathbf{Z}})$$

is the natural map.

Proof. Clearly $i^* \circ \pi^*$ is the identity endomorphism. Hence

$$\pi_* \circ \varphi \circ i_*(x) = \pi_* \circ \varphi \circ i_*(i^* \circ \pi^*(x)) = \pi_* \circ \varphi(i_*(1) \cup \pi^* x) = \pi_* \circ \varphi \circ i_*(1) \cup x$$

where $1 \in H^0(X, \mathcal{O}_X)$, from the projection formulas for π, φ and i. Thus we are reduce to showing that $\pi_* \circ \varphi \circ i_*(1) = 1$. From lemma 3, we eay replace X by its generic point. Hence we are reduced to the special case when π is the structure morphis $\pi: \mathbf{P}^m_K \to \operatorname{Spec} K$, and i: $\operatorname{Spec} K \to \mathbf{P}^m_K$ is the inclusion of a (closed) K-rational point $P \in \mathbf{P}^m$

Now $\varphi \circ i_*(1)$ is the image of $\alpha_0(i) \in \operatorname{Ext}^m_{\mathbf{P}^m_K}(i_* \mathcal{O}_P, \Omega^m_{\mathbf{P}^m_K/\mathbf{Z}})$ in $H^m_P(\mathbf{P}^m_K, \Omega^m_{\mathbf{P}^m_K/\mathbf{Z}})$. Thus $\pi_* \circ \varphi \circ i$ (1) is the image of $\alpha_0(i)$ under the map

$$\operatorname{Exp}^m_{\mathbf{P}^m_K}(i_* \mathcal{O}_P, \Omega^m_{\mathbf{P}^m_K/\mathbf{Z}}) \to \operatorname{Ext}^m_{\mathbf{P}^m_K}(i_* \mathcal{O}_P, \Omega^m_{\mathbf{P}^m_K/K}) \to H^m(\mathbf{P}^m_K, \Omega^m_{\mathbf{P}^m_K/K}) \xrightarrow{\mathrm{tr}} K.$$

Let

$$0 \to \Omega^1_{\mathbf{P}^m_K/K}(1) \to V \otimes_K \mathcal{O}_{\mathbf{P}^m_K} \to \mathcal{O}_{\mathbf{P}^m_K}(1) \to 0$$

be the universal exact sequence, where V is a K-vector space of dimension $m+$ and let P correspond to the linear functional $l: V \to K$. Then we have identification

$$\Omega^m_{\mathbf{P}^m_K/K} \cong \overset{m+1}{\wedge} V \otimes_K \mathcal{O}_{\mathbf{P}^m_K}(-m-1),$$

$$i^* \Omega^1_{\mathbf{P}^m_K/K}(1) \cong (\ker l) \otimes_K \mathcal{O}_P,$$

and hence an isomorphism

$$\overset{m+1}{\wedge} V \otimes_K (\overset{m}{\wedge} \ker l)^* \otimes_K i^* \mathcal{O}_{\mathbf{P}^m_K}(-1) \cong \mathcal{O}_P.$$

Next, if $\mathcal{I}_P$ is the ideal sheaf of P, there is a natural surjection

$$(\ker l) \otimes_K \mathcal{O}_{\mathbf{P}^m_K}(-1) \to \mathcal{I}_P,$$

and hence a Koszul resolution for $\mathcal{O}_P$. Tensoring it with

$$\mathfrak{L} \cong \overset{m+1}{\wedge} V \otimes_K (\overset{m}{\wedge} \ker l)^* \otimes_K \mathcal{O}_{\mathbf{P}^M_K}(-1),$$

we obtain an exact sequence

$$0 \to \overset{m+1}{\wedge} V \otimes_K \mathcal{O}_{\mathbf{P}^m_K}(-m-1) \to \overset{m+1}{\wedge} V \otimes_K (\overset{m}{\wedge} \ker l)^*$$
$$\otimes_K \overset{m-1}{\wedge} \ker l \otimes_K \mathcal{O}_{\mathbf{P}^m_K}(-m) \to \cdots \to \overset{m+1}{\wedge} V \otimes_K (\overset{m}{\wedge} \ker l)^* \otimes_K (\ker l)$$
$$\otimes_K \mathcal{O}_{\mathbf{P}^m_K}(-2) \to \overset{m+1}{\wedge} V \otimes_K (\overset{m}{\wedge} \ker l)^* \otimes_K \mathcal{O}_{\mathbf{P}^m_K}(-1) \to i_* \mathcal{O}_P \to 0.$$

One verifies easily that this exact sequence represents the class of $\alpha_0(i)$ in $\operatorname{Ext}^m_{\mathbf{P}^m_K}(i_* \mathcal{O}_P, \Omega^m_{\mathbf{P}^m_K/K}$

On the other hand, $\mathrm{tr}^{-1}(1) \in H^m(\mathbf{P}^m_K, \Omega^m_{\mathbf{P}^m_K/K})$ is represented by the class of the (exac Koszul complex

$$0 \to \overset{m+1}{\wedge} V \otimes_K \mathcal{O}_{\mathbf{P}^m_K}(-m-1) \to \cdots \to V \otimes_K \mathcal{O}_{\mathbf{P}^m_K}(-1) \to \mathcal{O}_{\mathbf{P}^m_K} \to 0.$$

We need to show that $\alpha_0(i)\mapsto \mathrm{tr}^{-1}(1)$; this follows at once from the fact that we have surjections

$$\overset{i}{\wedge} V\to \overset{m+1}{\wedge} V\otimes_K(\overset{m}{\wedge}\ker l)^*\otimes_K \overset{i-1}{\wedge}\ker l$$

induced by l, which give rise to a diagram

$$\begin{tikzcd}
0 \arrow[r] & \overset{m+1}{\wedge} V\otimes_K\mathcal{O}_{\mathbf{P}^m_K}(-m-1) \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & V\otimes_K\mathcal{O}_{\mathbf{P}^m_K}(-1) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbf{P}^m_K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{m+1}{\wedge} V\otimes_K\mathcal{O}_{\mathbf{P}^m_K}(-m-1) \arrow[r] & \cdots \arrow[r] & \overset{m+1}{\wedge} V\otimes_K(\overset{m}{\wedge}\ker l)^*\otimes_K\mathcal{O}_{\mathbf{P}^m_K}(-1) \arrow[r] & i_*\mathcal{O}_P \arrow[r] & 0
\end{tikzcd}$$

(where the vertical arrow on the extreme left is the identity map on

$$\Omega^m_{\mathbf{P}^m_K/K}\cong \overset{m+1}{\wedge} V\otimes_K\mathcal{O}_{\mathbf{P}^m_K}(-m-1),$$

and that on the extreme right is the natural map $\mathcal{O}_{\mathbf{P}^m_K}\to i_*\mathcal{O}_P$; thus the top row represents the same class in Ext^m as does the pullback of the bottom row under $\mathcal{O}_{\mathbf{P}^m_K}\to i_*\mathcal{O}_P$).

Lemma 9. *Let $i:X\to Y$ be an admissible closed embedding, and let $\mathscr{E}$ be a locally free $\mathcal{O}_Y$-module of rank $m+1$. Let $i':X\to\mathbf{P}(\mathscr{E})$ be a morphism (necessarily an admissible closed embedding) such that the following diagram commutes:*

$$\begin{tikzcd}
X \arrow[r, "i'"] \arrow[dr, "i"'] & \mathbf{P}_Y(\mathscr{E}) \arrow[d, "\pi"] \\
& Y
\end{tikzcd}$$

Then $i_*=\pi_*\circ\theta\circ i'_*:H^p(X,\Omega^q_{X/\mathbf{Z}})\to H^{p+d}_X(Y,\Omega^{q+d}_{Y/\mathbf{Z}})$, where $d=\dim Y-\dim X$, and

$$\theta:H^{p+d+m}_{i'(X)}(\mathbf{P}_Y(\mathscr{E}),\Omega^{q+d+m}_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}})\to H^{p+d+m}_{\pi^{-1}(i(X))}(\mathbf{P}_Y(\mathscr{E}),\Omega^{q+d+m}_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}})$$

is the natural map.

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathbf{P}_X(i'^*\mathscr{E}) \arrow[r, "j''"] \arrow[d, shift left, "\pi'"] & \mathbf{P}_Y(\mathscr{E}) \arrow[d, "\pi"] \\
X \arrow[r, "i"] \arrow[u, shift left, "j'^*"] & Y
\end{tikzcd}$$

where j', j'' are closed embeddings, $j''\circ j'=i'$ and j' is a section of π'. If

$$\psi:H^{p+m}_{j'(X)}(\mathbf{P}_X(i'^*\mathscr{E}),\Omega^{q+m}_{\mathbf{P}_X(i'^*\mathscr{E})/\mathbf{Z}})\to H^{p+m}(\mathbf{P}_X(i'^*\mathscr{E}),\Omega^{q+m}_{\mathbf{P}_X(i'^*\mathscr{E})/\mathbf{Z}})$$

is the natural map, then from lemma 2, we see that $\theta\circ i'_*=j''_*\circ\psi\circ j'_*$. Next, from lemma 7, $\pi_*\circ j''_*=i_*\circ\pi'_*$, and from lemma 8, $\pi'_*\circ\psi\circ j'_*$ is the identity map. We now compute:

$$\pi_* \circ \theta \circ i'_* = \pi_* \circ j''_* \circ \psi \circ j'_* = i_* \circ \pi'_* \circ \psi \circ j'_* = i_*. \qquad \square$$

Lemma 10. *Let Y be a Noetherian separated integral regular scheme such that $\Omega^1_{Y/Z}$ is locally free. Let $\mathscr{E}$, $\mathscr{F}$ be locally free sheaves on Y of ranks $m+1$, $n+1$ respectively, and let $X = \mathbf{P}_Y(\mathscr{E}) \times_Y \mathbf{P}_Y(\mathscr{F})$, so that we have a diagram of admissible morphisms*

$$\begin{tikzcd}
X \arrow[r, "\pi'"] \arrow[d, "\eta'"'] \arrow[dr, "f"] & \mathbf{P}_Y(\mathscr{F}) \arrow[d, "\eta"] \\
\mathbf{P}_Y(\mathscr{E}) \arrow[r, "\pi"] & Y
\end{tikzcd}$$

Let $Z \subset Y$ be a closed subscheme, $V = \pi^{-1}(Z)$, $W = \eta^{-1}(Z)$, $T = f^{-1}(Z)$. Then for any $p, q \geqslant m+n$, we have

$$\pi_* \circ \eta'_* = \eta_* \circ \pi_* : H^p_T(X, \Omega^q_{X/Z}) \to H^{p-m-n}_Z(Y, \Omega^{q-m-n}_{Y/Z})$$

(note that $X = \mathbf{P}_{\mathbf{P}_Y(\mathscr{E})}(\pi^* \mathscr{F}) \cong \mathbf{P}_{\mathbf{P}_Y(\mathscr{F})}(\eta^* \mathscr{E})$, so π'_*, η'_* exist).

Proof. One shows quite easily that both composites equal the following composite map:

$$\begin{aligned}
H^p_T(X, \Omega^q_{X/Z}) &\to H^p_T(X, \Omega^{m+n}_{X/Y} \otimes f^* \Omega^{q-m-n}_{Y/Z}) \\
&\xrightarrow{\text{edge}} H^{p-m-n}_Z(Y, R^{m+n} f_* \Omega^{m+n}_{X/Y} \otimes \Omega^{q-m-n}_{Y/Z}) \\
&\cong H^{p-m-n}_Z(Y, \Omega^{q-m-n}_{Y/Z}).
\end{aligned}$$

This uses the isomorphisms

$$\Omega^n_{X/\mathbf{P}_Y(\mathscr{E})} \otimes \eta'^* \Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \cong \Omega^{m+n}_{X/Z} \cong \Omega^m_{X/\mathbf{P}_Y(\mathscr{F})} \otimes \pi'^* \Omega^n_{\mathbf{P}_Y(\mathscr{F})/Y},$$

and the resulting isomorphisms

$$\begin{aligned}
\pi_* R^n \eta'_* \Omega^n_{X/\mathbf{P}_Y(\mathscr{E})} \otimes R^m \pi_* \Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} &\cong R^n \eta_* \Omega^n_{\mathbf{P}_Y(\mathscr{F})/Y} \otimes R^m \pi_* \Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \\
&\cong R^{m+n} f_* \Omega^{m+n}_{X/Y} \cong \\
\eta_* R^m \pi'_* \Omega^m_{X/\mathbf{P}_Y(\mathscr{F})} \otimes R^n \eta_* \Omega^n_{\mathbf{P}_Y(\mathscr{F})/Y} &\cong R^m \pi_* \Omega^m_{\mathbf{P}_Y(\mathscr{E})/Y} \otimes R^n \eta_* \Omega^n_{\mathbf{P}_Y(\mathscr{F})/Y}
\end{aligned}$$

(note that inverting the order in the tensor product $R^m \pi_* \Omega^m \otimes R^n \eta_* \Omega^n$ introduces a sign of $(-1)^{2m+2n} = 1$). $\square$

We can now (finally!) construct the Gysin maps for an arbitrary admissible morphism $f: X \to Y$. Choose some factorisation

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[dr, "f"'] & \mathbf{P}_Y(\mathscr{E}) \arrow[d, "\pi"] \\
 & Y
\end{tikzcd}$$

where $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, and i is a closed embedding; then π, i are admissible. Let $d = \dim Y - \dim X$, $m + 1 = \operatorname{rank} \mathcal{E}$, and let

$$\theta = \theta_{p,q}: H^{p+d+m}_{i(X)}(\mathbf{P}_Y(\mathcal{E}), \Omega^{q+d+m}_{\mathbf{P}_Y(\mathcal{E})/Z}) \to H^{p+d+m}_{\pi^{-1}(f(X))}(\mathbf{P}_Y(\mathcal{E}), \Omega^{q+d+m}_{\mathbf{P}_Y(\mathcal{E})/Z});$$

define

$$f_* = \pi_* \circ \theta \circ i_*: H^p(X, \Omega^q_{X/Z}) \to H^{p+d}_{f(X)}(Y, \Omega^{q+d}_{Y/Z}).$$

The main result of this section is the following.

Theorem 1. (*a*) *Let $f: X \to Y$ be an admissible morphism. Then the Gysin maps f_* are independent of the factorisation $f = \pi \circ i$, used above in defining f_*.*
(*b*) *If $f: X \to Y$, $g: Y \to Z$ are admissible morphisms, then $g \circ f$ is admissible; if*

$$\psi = \psi_{r,s}: H^r_{f(X)}(Y, \Omega^s_{Y/Z}) \to H^r(Y, \Omega^s_{Y/Z}),$$

$$\theta = \theta_{r,s}: H^r_{g\circ f(X)}(Z, \Omega^s_{Z/Z}) \to H^r_{g(Y)}(Z, \Omega^s_{Z/Z})$$

are the natural maps, then $g_* \circ \psi \circ f_* = \theta \circ (g \circ f)_*$ on $H^p(X, \Omega^q_{X/Z})$ for all $p, q \geqslant 0$.
(*c*) *If*

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

is a fibre square, where f, f' are admissible, and g, h are flat, then $g^* \circ f_* = f'_* \circ h^*$ on $H^p(X, \Omega^q_{X/Z})$ for all $p, q \geqslant 0$.
(*d*) *If $f: X \to Y$ is admissible, then f_* satisfies the projection formula, i.e., for all $x \in H^p(X, \Omega^q_{X/Z})$, $y \in H^r(Y, \Omega^s_{Y/Z})$, we have*

$$f_*(x \cup f^* y) = f_* x \cup y \in H^{p+r+d}_{f(X)}(Y, \Omega^{q+s+d}_{Y/Z}),$$

where $d = \dim Y - \dim X$.
(*e*) *If $f: X \to Y$ is an admissible morphism between smooth varieties over a field k, then the following diagram commutes:*

$$\begin{tikzcd} H^p(X, \Omega^q_{X/Z}) \arrow[r] \arrow[d, "f_*"'] & H^p(X, \Omega^q_{X/k}) \arrow[d, "f_*"] \\ H^{p+d}_{f(X)}(Y, \Omega^{q+d}_{Y/Z}) \arrow[r] & H^{p+d}_{f(X)}(Y, \Omega^{q+d}_{Y/k}) \end{tikzcd}$$

Here the horizontal maps are the natural ones, induced by the sheaf maps $\Omega^1_{X/Z} \to \Omega^1_{X/k}$, $\Omega^1_{Y/Z} \to \Omega^1_{Y/k}$; then vertical arrow on the right is the standard Gysin map given by Grothendieck duality (*again*, $d = \dim Y - \dim X$).
(*f*) *If $f: X \to Y$ is a surjective, generically finite admissible morphism, then there are unique trace maps*

$$\operatorname{tr}_{X/Y}: f_* \Omega^q_{X/Z} \to \Omega^q_{Y/Z}$$

with the following properties. Firstly, $\operatorname{tr}_{X/Y}$ is the usual trace (*induced by the trace on* *function fields) for* $q = 0$. *Next,* $\mathrm{tr}_{X/Y}$ *satisfies*

$$\mathrm{tr}_{X/Y}(\omega \wedge f^*\eta) = \mathrm{tr}_{X/Y}(\omega) \wedge \eta.$$

Finally,

$$f_*: H^0(X, \Omega^q_{X/\mathbf{Z}}) \to H^0(Y, \Omega^q_{Y/\mathbf{Z}})$$

satisfies $f_*\omega = \mathrm{tr}_{X/Y}(\omega)$ *for all* $q \geqslant 0$.

(*g*) *If* $i: X \to Y$ *is an admissible closed embedding, such that* $i(X)$ *is a Cartier divisor in* Y, *then*

$$i_*: H^0(X, \mathcal{O}_X) \to H^1_{i(X)}(Y, \Omega^1_{Y/\mathbf{Z}})$$

satisfies $i_*1 = c_1(\mathcal{O}_Y(i(X))$. *Here* $c_1(\mathcal{O}_Y(i(X))$ *is the "localised" first Chern class, i.e., the image of the standard generator of* $H^1_{i(X)}(Y, \mathcal{O}^*_Y) = \mathbf{Z}$ *under the map*

$$\mathrm{dlog}: H^1_{i(X)}(Y, \mathcal{O}^*_Y) \to H^1_{i(X)}(Y, \Omega^1_{Y/\mathbf{Z}}).$$

Proof[1]

(a) Suppose

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & \mathbf{P}_Y(\mathscr{E}) \arrow[d, "\pi"] \\ & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & \mathbf{P}_Y(\mathscr{F}) \arrow[d, "\eta"] \\ & Y \end{tikzcd}$$

are two factorisations of f. Let $k: X \to \mathbf{P}_Y(\mathscr{E}) \times_Y \mathbf{P}_Y(\mathscr{F})$ be the resulting embedding, and consider the diagram

$$\begin{tikzcd} \mathbf{P}_Y(\mathscr{E}) \times \mathbf{P}_Y(\mathscr{F}) \arrow[r, "\pi'"] \arrow[d, "\eta'"'] \arrow[dr, "g"] & \mathbf{P}_Y(\mathscr{F}) \arrow[d, "\eta"] \\ \mathbf{P}_Y(\mathscr{E}) \arrow[r, "\pi"] & Y \end{tikzcd}$$

By lemma 10, $\pi_* \circ \eta'_* = \eta_* \circ \pi'_*$; let g_* denote either of these. Let $T = \mathbf{P}_Y(\mathscr{E}) \times_Y \mathbf{P}_Y(\mathscr{F})$. From lemmas 9 and 10, we have a commutative diagram

$$\begin{tikzcd} & H^p(X, \Omega^q_{X/\mathbf{Z}}) \arrow[dl, "i_*"'] \arrow[d, "k_*"] \arrow[dr, "j_*"] & \\ H^{p+d+m}_{i(X)}(\mathbf{P}_Y(\mathscr{E}), \Omega^{q+d+m}_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}}) \arrow[d] & H^{p+d+m+n}_{k(X)}(T, \Omega^{q+d+m+n}_{T/\mathbf{Z}}) \arrow[l] \arrow[r] \arrow[d] & H^{p+d+n}_{j(X)}(\mathbf{P}_Y(\mathscr{F}), \Omega^{q+d+n}_{\mathbf{P}_Y(\mathscr{F})/\mathbf{Z}}) \arrow[d] \\ H^{p+d+m}_{\pi^{-1}f(X)}(\mathbf{P}_Y(\mathscr{E}), \Omega^{q+d+m}_{\mathbf{P}_Y(\mathscr{E})/\mathbf{Z}}) \arrow[dr, "\pi_*"'] & H^{p+d+m+n}_{g^{-1}f(X)}(T, \Omega^{q+d+m+n}_{T/\mathbf{Z}}) \arrow[l, "\eta'_*"'] \arrow[r, "\pi'_*"] \arrow[d, "g_*"] & H^{p+d+n}_{\eta^{-1}f(X)}(\mathbf{P}_Y(\mathscr{F}), \Omega^{q+d+n}_{\mathbf{P}_Y(\mathscr{F})/\mathbf{Z}}) \arrow[dl, "\eta_*"] \\ & H^{p+d}_{f(X)}(Y, \Omega^{q+d}_{Y/\mathbf{Z}}) & \end{tikzcd}$$

In particular, the maps $H^p(X, \Omega^q_{X/\mathbf{Z}}) \to H^{p+d}_{f(X)}(Y, \Omega^{q+d}_{Y/\mathbf{Z}})$ given by the left and right borders agree, i.e., f_*, defined using either factorisation, is the same.

---

[1] At a first reading, the reader may find it simpler to ignore all conditions on the supports.

(b) It is easy to see that $g \circ f$ is admissible. Choose factorisations of g and $g \circ f$,

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[dr, "g\circ f"'] & \mathbf{P}_Z(\mathscr{E}) \arrow[d, "\pi"] \\
& Z
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "j"] \arrow[dr, "g"'] & \mathbf{P}_Z(\mathscr{F}) \arrow[d, "\eta"] \\
& Z
\end{tikzcd}$$

so that f has an induced factorisation

$$\begin{tikzcd}
X \arrow[r, "i''"] \arrow[dr, "f"'] & \mathbf{P}_Y(g^*\mathscr{E}) \arrow[d, "\pi''"] \\
& Y
\end{tikzcd}$$

where $i'' = (f, i): X \to Y \times_Z \mathbf{P}_Z(\mathscr{E}) = \mathbf{P}_Y(g^*\mathscr{E})$. Suppose rank $\mathscr{E} = m+1$, rank $\mathscr{F} = n+1$. Let $T = \mathbf{P}_Y(\mathscr{E}) \times_Y \mathbf{P}_Y(\mathscr{F})$; if $g': \mathbf{P}_Y(g^*\mathscr{E}) \to \mathbf{P}_Z(\mathscr{E})$ is the morphism induced by g, then $i' = (g', j \circ \pi''): \mathbf{P}_Y(g^*\mathscr{E}) \to T$, so that $i = \eta' \circ i' \circ i''$. We have the following diagram:

$$\begin{tikzcd}
X \arrow[r, "i''"] \arrow[dr, "f"'] & \mathbf{P}_Y(g^*\mathscr{E}) \arrow[r, "i'"] \arrow[d, "\pi''"] & T \arrow[r, "\eta'"] \arrow[d, "\pi'"] & \mathbf{P}_Z(\mathscr{E}) \arrow[ddl, "\pi"] \\
& Y \arrow[r, "j"] \arrow[dr, "g"'] & \mathbf{P}_Z(\mathscr{F}) \arrow[d, "\eta"'] & \\
& & Z &
\end{tikzcd}$$

By lemmas 9 and 10, we have a factorisation

$$\theta \circ (g \circ f)_* = \theta \circ \eta_* \circ \pi'_* \circ \tilde{\theta} \circ (i' \circ i'')_*$$

where

$$\tilde{\theta} = \tilde{\theta}_{s,t}: H^s_{i' \circ i''(X)}(T, \Omega^t_{T/Z}) \to H^s_{(\eta \circ \pi')^{-1}(g \circ f(X))}(T, \Omega^t_{T/Z}).$$

From lemma 2, we have $(i' \circ i'')_* = i'_* \circ \tilde{\psi} \circ i''_*$ where $\tilde{\psi} = \tilde{\psi}_{s,t}$ denotes the operation "forget supports" on $\mathbf{P}_Y(g^*\mathscr{E})$ (analogous to ψ on Y). If $\bar{\theta}$ is the analogue of $\tilde{\theta}$ on $\mathbf{P}_Z(\mathscr{F})$ (i.e., $\bar{\theta}$ is the operation "enlarge the supports to $\eta^{-1}(g \circ f(X))$") then $\bar{\theta} \circ \pi'_* = \pi'_* \circ \tilde{\theta}$, by lemma 5. Thus

$$\begin{aligned}
\theta \circ (g \circ f)_* &= \theta \circ \eta_* \circ \bar{\theta} \circ \pi'_* \circ i'_* \circ \tilde{\psi} \circ i''_* \\
&= \theta \circ \eta_* \circ \bar{\theta} \circ j_* \circ \pi''_* \circ \tilde{\psi} \circ i''_* \quad \text{(by lemma 7)} \\
&= \eta_* \circ \theta' \circ j_* \circ \psi \circ \pi''_* \circ \theta'' \circ i''_*
\end{aligned}$$

where θ' is the operation "increase the support to $\eta^{-1} g(Y)$" and θ'' is the operation "increase the support to $\pi''^{-1} f(X)$". But $f_* = \pi''_* \circ \theta'' \circ i''_*$, and $g_* = \eta_* \circ \theta' \circ j_*$. This proves (b).

(c) This follows from lemma 3 and the compatibility of the Gysin maps for $\pi: \mathbf{P}_Y(\mathscr{E}) \to Y$ with an arbitrary pullback $g: Y' \to Y$.

(d) This is immediate from lemmas 4 and 6, and the fact that the operation "increase supports" is compatible with cup products.

(e) This is easy to see from our definitions, and standard computations of the Gysin maps, coming from Grothendieck duality, for a closed embedding of smooth schemes, and for the structure morphism of a projective bundle (see [H2], appendix by P. Deligne). We omit the details.

(f) The requirement that $f_*: H^0(X, \Omega^q_{X/\mathbf{Z}}) \to H^0(Y, \Omega^q_{Y/\mathbf{Z}})$ is given by the trace, and the compatibility of the Gysin maps with flat base change (in particular with the base change given by restriction to an open set $U \subset Y$), uniquely defines maps of sheaves

$$\mathrm{tr}_{X/Y} \Omega^q_{X/\mathbf{Z}} \to \Omega^q_{Y/\mathbf{Z}}.$$

The formula

$$\mathrm{tr}_{X/Y}(\omega \wedge f^*\eta) = \mathrm{tr}_{X/Y}(\omega) \wedge \eta$$

is then a particular case of the projection formula. We have only to verify that for $q = 0$, this reduces to the usual trace on functions. Again, by compatibility with flat base change, we are reduced to checking the result when $X = \operatorname{Spec} L$, $Y = \operatorname{Spec} K$ and L/K is a finite extension of fields. Then $f: X \to Y$ has a factorisation

$$\begin{tikzcd}
\operatorname{Spec} L = X \arrow[r, "i"] \arrow[dr, "f"'] & \mathbf{P}^1_K \arrow[d, "\pi"] \\
& Y = \operatorname{Spec} K
\end{tikzcd}$$

where $i(X) = Q \in \mathbf{P}^1_K$ is a closed point with residue field L. Now $f_*: L \to K$ is the composite

$$L = H^0(X, \mathcal{O}_X) \xrightarrow{i_*} H^1_Q(\mathbf{P}^1_K, \Omega^1_{\mathbf{P}^1_K/K}) \xrightarrow{\mathrm{tr}} H^0(Y, \mathcal{O}_Y) = K$$

where $\mathrm{tr}: H^1(\mathbf{P}^1, \Omega^1_{\mathbf{P}^1_K/K}) \to H^0(Y, \mathcal{O}_Y)$ is the trace map of Grothendieck duality for π. From the resolution

$$0 \to \mathcal{O}_{\mathbf{P}^1_K}(-Q) \to \mathcal{O}_{\mathbf{P}^1_K} \to i_*\mathcal{O}_Q \to 0$$

we have

$$\begin{aligned}
\mathrm{Ext}^1_{\mathbf{P}^1_K}(i_*\mathcal{O}_Q, \Omega^1_{\mathbf{P}^1_K/K}) &\cong \mathrm{Hom}_Q(i^*\mathcal{O}_{\mathbf{P}^1_K}(-Q), i^*\Omega^1_{\mathbf{P}^1_K/K}) \\
&\cong H^0(Q, i^*\Omega^1_{\mathbf{P}^1_K/K} \otimes i^*\mathcal{O}_{\mathbf{P}^1_K}(Q)).
\end{aligned}$$

The canonical generator $\alpha_0(i)$ of this Ext group corresponds to the section $i^*(dt/t)$, where $t \in \mathcal{O}_{Q,\mathbf{P}^1_K}$ is a local parameter (this section is independent of the choice of t). The map $f_*: L \to K$ is given by

$$\alpha \mapsto \mathrm{Res}_Q\left(a\frac{dt}{t}\right) \in K$$

where Res_Q is the residue symbol (see [T], for example). However,

$$\mathrm{Res}_Q\left(a\frac{dt}{t}\right) = \mathrm{tr}_{L/K}(a)$$

(see *loc. cit.*).

(g) As in the proof of (f) above, we see that

$$\mathrm{Ext}^1_Y(i_*\mathcal{O}_X, \Omega^1_{Y/\mathbf{Z}}) \cong H^0(X, i^*\Omega^1_{Y/\mathbf{Z}}(i(X)),$$

and if t is a local generator of the ideal sheaf of $i(X) \subset Y$, then $i^*(dt/t)$ is a local expression for $\alpha_0(i)$. Hence, we must show that the corresponding element of $H^1_{i(X)}(Y, \Omega^1_{Y/\mathbf{Z}})$ is the image of the standard generator $1 \in H^1_{i(X)}(Y, \mathcal{O}^*_Y)$.

Let $\eta \in Y$ be the generic point, and let Y^1 be the set of codimension 1 points of Y. If $y \in Y$, and A is an abelian group, let $i_{y*}A$ denote the constant sheaf determined by A on $\overline{\{y\}}$. Then we have a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}^*_Y \arrow[r] \arrow[d, "\mathrm{dlog}"'] & i_{\eta*}k(\eta)^* \arrow[r] \arrow[d, "\mathrm{dlog}"'] & \bigoplus_{y\in Y^1} i_{y*}\mathbf{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_{Y/\mathbf{Z}} \arrow[r] & i_{\eta*}\Omega^1_{k(\eta)/\mathbf{Z}} \arrow[r] & \bigoplus_{y\in Y^1} i_{y*}(\Omega^1_{k(\eta)/\mathbf{Z}}/\Omega^1_{\mathcal{O}_{y,Y/\mathbf{Z}}}) \arrow[r] & 0
\end{tikzcd}$$

where for $y \in Y$, $k(y)$ denotes the residue field at y. The rows give flasque resolutions, which yield a diagram

$$\begin{tikzcd}
\mathbf{Z} \arrow[r, "\cong"] \arrow[d, "\Phi"'] & H^1_{i(X)}(Y, \mathcal{O}^*_Y) \arrow[d] \\
(\Omega^1_{k(\eta)/\mathbf{Z}}/\Omega^1_{\mathcal{O}_{x,Y}/\mathbf{Z}}) \arrow[r, "\cong"] & H^1_{i(X)}(Y, \Omega^1_{Y/\mathbf{Z}})
\end{tikzcd}$$

where $x \in Y^1$ is the generic point of $i(X)$, and the horizontal arrows are isomorphisms.

We also have a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_{Y/\mathbf{Z}} \arrow[r] \arrow[d, equal] & \Omega^1_{Y/\mathbf{Z}} \otimes \mathcal{O}_Y(i(X)) \arrow[r] \arrow[d] & i_*i^*(\Omega^1_{Y/\mathbf{Z}} \otimes \mathcal{O}_Y(i(X))) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_{Y/\mathbf{Z}} \arrow[r] & i_{\eta*}\Omega^1_{k(\eta)/\mathbf{Z}} \arrow[r] & \bigoplus_{y\in Y^1} i_{y*}\Omega^1_{k(\eta)/\mathbf{Z}}/\Omega^1_{\mathcal{O}_{y,Y}/\mathbf{Z}} \arrow[r] & 0
\end{tikzcd}$$

We have an identification

$$\begin{aligned}
\mathrm{Ext}^1_{\mathcal{O}_Y}(i_*\mathcal{O}_X, \Omega^1_{Y/\mathbf{Z}}) &\cong \mathrm{Hom}_X(i^*(\mathscr{I}/\mathscr{I}^2), i^*\Omega^1_{Y/\mathbf{Z}}) \\
&= \mathrm{Hom}_X(i^*\mathcal{O}_Y(-i(X)), i^*\Omega^1_{Y/\mathbf{Z}}) \\
&\cong \Gamma(X, i^*(\Omega^1_{Y/\mathbf{Z}} \otimes \mathcal{O}_Y(i(X)))) \cong \Gamma_X(Y, i_*i^*(\Omega^1_{Y/\mathbf{Z}} \otimes \mathcal{O}_Y(i(X)))).
\end{aligned}$$

The natural map $\mathrm{Ext}^1_{\mathcal{O}_Y}(i_*\mathcal{O}_X, \Omega^1_{Y/\mathbf{Z}}) \to H^1_{i(X)}(Y, \Omega^1_{Y/\mathbf{Z}})$ corresponds to the map

$$\Gamma_X(Y, i_*i^*(\Omega^1_{Y/\mathbf{Z}} \otimes \mathcal{O}_Y(i(X)))) \to \Omega^1_{k(\eta)/\mathbf{Z}}/\Omega^1_{\mathcal{O}_{x,Y}/\mathbf{Z}}$$

which factors as

$$\begin{tikzcd}
\Gamma_X(Y, i_*i^*(\Omega^1_{Y/\mathbf{Z}} \otimes \mathcal{O}_Y(i(X)))) \arrow[rr] \arrow[dr, "\mu"'] & & \Omega^1_{k(\eta)/\mathbf{Z}}/\Omega^1_{\mathcal{O}_{x,Y}/\mathbf{Z}} \\
& \Omega^1_{\mathcal{O}_{x,Y}/\mathbf{Z}} \cdot t^{-1}/\Omega^1_{\mathcal{O}_{x,Y}/\mathbf{Z}} \arrow[ur] &
\end{tikzcd}$$

where $t \in \mathcal{O}_{x,Y}$ is a local parameter. Under the isomorphism

$$\mathrm{Hom}_X(i^*\mathcal{O}_Y(-i(X)), i^*\Omega^1_{Y/\mathbf{Z}}) \cong \Gamma(X, i^*(\Omega^1_{Y/\mathbf{Z}} \otimes \mathcal{O}_Y(i(X)))$$

the natural map

$$i^*\mathcal{O}_Y(-i(X)) \to i^*\Omega^1_{Y/\mathbf{Z}}$$

(which corresponds to $\alpha_0(i) \in \mathrm{Ext}^1_Y(i_*\mathcal{O}_X, \Omega^1_{Y/\mathbf{Z}})$) yields a section whose image (under μ) in

$$\Omega^1_{\mathcal{O}_{x,Y}/\mathbf{Z}} t^{-1}/\Omega^1_{\mathcal{O}_{x,Y}/\mathbf{Z}}$$

is the class of (dt/t) (this class is independent of the choice of the local parameter t). But the map

$$\Phi: \mathbf{Z} \to \Omega^1_{k(\eta)/\mathbf{Z}}/\Omega^1_{\mathcal{O}_{x,Y}/\mathbf{Z}}$$

is the composite

$$\mathbf{Z} \xrightarrow{v^{-1}} k(\eta)^*/\mathcal{O}^*_{x,Y} \xrightarrow{\mathrm{dlog}} \Omega^1_{k(\eta)/\mathbf{Z}}/\Omega^1_{\mathcal{O}_{x,Y}/\mathbf{Z}}$$

where $v \in k(\eta)^*/\mathcal{O}^*_{x,Y} \to \mathbf{Z}$ is the discrete valuation corresponding to x. Clearly $v^{-1}(1) = t (\mathrm{mod}\, \mathcal{O}^*_{x,Y})$, so that

$$\Phi(1) = \frac{dt}{t} (\mathrm{mod}\, \Omega^1_{\mathcal{O}_{x,Y}/\mathbf{Z}}).$$

2. Cycle classes

Let X be an integral scheme of finite type over a field k, and let $S \subset X$ be the closed subset of non-regular points. The *relative Chow groups* $CH^i(X,S)$ are defined as follows.

Let $\mathscr{Z}^i(X,S)$ (the group of *relative cycles* of codimension i) be the free abelian group on the set of closed integral subschemes $Z \subset X$ of codimension i such that $Z \cap S = \phi$. Let $\mathscr{R}^i(X,S) \subset \mathscr{Z}^i(X,S)$ be the subgroup generated by cycles of the form $(f)_W$, where $W \subset X$ is a closed integral subscheme of codimension $i-1$ such that $W \cap S = \phi$, and f is a rational function on W. Here,

$$(f)_W = \sum n_i [Z_i]$$

where Z_i runs over the closed integral subschemes of codimension 1 in W, and

$$n_i = \mathrm{mult}_{Z_i}(f) = l(A_i/a_iA_i) - l(A_i/b_iA_i),$$

where $A_i = \mathcal{O}_{Z_i W}$, and $f = a_i/b_i$ with $a_i, b_i \in A_i$ (and $l(M)$ denotes the length of M). We define

$$CH^i(X,S) = \mathscr{Z}^i(X,S)/\mathscr{R}^i(X,S).$$

Our goal in this section is to define a *cycle class* homomorphism

$$Cl_X: CH^i(X,S) \to H^i(X, \Omega^i_{X/\mathbf{Z}}).$$

Let $Z \subset X$ be a closed integral subscheme of codimension i with $Z \cap S = \phi$. Let $\varphi : \tilde{Z} \to Z$ be the normalisation; let $T = \varphi(\tilde{Z}_{\text{sing}})$, where $\tilde{Z}_{\text{sing}} \subset \tilde{Z}$ is the closed subset of non-regular points. Then

$$\varphi : \tilde{Z} - \varphi^{-1}(T) \to X - (S \cup T)$$

is a finite morphism between integral regular k-schemes of finite type, and is hence admissible (see lemma 1). Thus, by Theorem 1, we have a map

$$\varphi_* : H^0(\tilde{Z} - \varphi^{-1}(T), \mathcal{O}_{\tilde{Z}}) \to H^i_{Z-T}(X - (S \cup T), \Omega^i_{X/\mathbf{Z}}).$$

Now $X - (S \cup T)$ is an open neighbourhood in $X - T$ of the closed subscheme $Z - T$. Hence

$$H^i_{Z-T}(X - (S \cup T), \Omega^i_{X/\mathbf{Z}}) \cong H^i_{Z-T}(X - T, \Omega^i_{X/\mathbf{Z}}).$$

by excision for local cohomology. Next, $\operatorname{codim}_X T \geqslant i + 2$, and $\Omega^i_{X/\mathbf{Z}}$ is locally free in a neighbourhood of T; hence

$$H^i_T(X, \Omega^i_{X/\mathbf{Z}}) \cong H^{i+1}_T(X, \Omega^i_{X/\mathbf{Z}}) = 0.$$

Hence

$$H^i_Z(X, \Omega^i_{X/\mathbf{Z}}) \cong H^i_{Z-T}(X - T, \Omega^i_{X/\mathbf{Z}}).$$

We define $Cl_X(Z)$ to be the image of

$$1 \in H^0(\tilde{Z} - \varphi^{-1}(T), \mathcal{O}_{\tilde{Z}})$$

under the composite map

$$\begin{aligned} H^0(\tilde{Z} - \varphi^{-1}(T), \mathcal{O}_{\tilde{Z}}) &\xrightarrow{\phi_*} H^i_{Z-T}(X - (S \cup T), \Omega^i_{X/\mathbf{Z}}) \\ &\cong H^i_Z(X, \Omega^i_{X/\mathbf{Z}}) \\ &\to H^i(X, \Omega^i_{X/\mathbf{Z}}). \end{aligned}$$

If $Z_1 \subset Z$ is the singular locus, then $\operatorname{codim}_X Z_1 \geqslant i + 1$, so $H^i_{Z_1}(X, \Omega^i_{X/\mathbf{Z}}) = 0$. Hence

$$H^i(X, \Omega^i_{X/\mathbf{Z}}) \xrightarrow{\xi} H^i(X - Z_1, \Omega^i_{X/\mathbf{Z}})$$

is injective. (In fact

$$H^i_Z(X, \Omega^i_{X/\mathbf{Z}}) \to H^i_{Z-Z_1}(X - Z_1, \Omega^i_{X/\mathbf{Z}})$$

is also injective, though we do not need this below.) Let $j : Z - Z_1 \hookrightarrow X - Z_1$ be the inclusion. By Theorem 1 (c), there is a commutative diagram

$$\begin{tikzcd}
H^0(\tilde{Z} - \varphi^{-1}(T), \mathcal{O}_{\tilde{Z}}) \arrow[r] \arrow[d, "\varphi_*"] & H^0(\tilde{Z} - \varphi^{-1}(Z_1), \mathcal{O}_{\tilde{Z}}) \cong & H^0(Z - Z_1, \mathcal{O}_Z) \arrow[d, "j_*"] \\
H^i_{Z-T}(X - T, \Omega^i_{X/\mathbf{Z}}) \arrow[rr] \arrow[d, "\wr"] & & H^i_{Z-Z_1}(X - Z_1, \Omega^i_{X/\mathbf{Z}}) \arrow[d, equal] \\
H^i_Z(X, \Omega^i_{X/\mathbf{Z}}) \arrow[rr] \arrow[d] & & H^i_{Z-Z_1}(X - Z_1, \Omega^i_{X/\mathbf{Z}}) \arrow[d] \\
H^i(X, \Omega^i_{X/\mathbf{Z}}) \arrow[rr, "\xi"] & & H^i(X - Z_1, \Omega^i_{X/\mathbf{Z}})
\end{tikzcd}$$

Here, as we noted above, ξ is injective. Hence $Cl_X(Z)$ is the unique element of $H^i(X, \Omega^i_{X/Z})$ whose image in $H^0(X - Z_1, \Omega^i_{X/Z})$ is the image of $1 \in H^0(Z - Z_1, \mathcal{O}_Z)$ under

$$H^0(Z - Z_1, \mathcal{O}_Z) \xrightarrow{j_*} H^i_{Z-Z_1}(X - Z_1, \Omega^i_{X/Z}) \to H^i(X - Z_1, \Omega^i_{X/Z}).$$

Theorem 2. *The assignment $Z \mapsto Cl_X(Z)$ induces a well defined homomorphism*

$$Cl_X : CH^i(X, S) \to H^i(X, \Omega^i_{X/Z}).$$

Proof. Let $W \subset X$ be an integral closed subscheme of codimension $i-1$ with $W \cap S = \phi$, and let f be a rational function on W, with

$$(f)_W = \sum_{p=1}^{r} n_p [Z_p]$$

where the $Z_p \subset W$ are precisely the integral closed subschemes $Z \subset W$ of codimension 1 such that $f \notin \mathcal{O}^*_{Z,W}$. We must show that

$$\sum_{p=1}^{r} n_p Cl_X(Z_p) = 0$$

in $H^i(X, \Omega^i_{X/Z})$.

Let $\psi : \tilde{W} \to W$ be the normalisation. Let

$$\psi^{-1}(Z_p) = \cup Z_{pq}$$

be the decomposition into irreducible components; let

$$Z'_{pq} = \begin{cases} Z_{pq} & \text{if } Z_{pq} \to Z_p \text{ is } \textit{not} \text{ surjective;} \\ (Z_{pq})_{\text{sing}} & \text{otherwise.} \end{cases}$$

Set

$$V = \psi(\tilde{W}_{\text{sing}}) \cup (\cup_p (Z_p)_{\text{sing}}) \cup (\cup_{p,q} \psi(Z'_{pq})).$$

Then codim $_X V \geq i+1$, so that

$$H^i(X, \Omega^i_{X/Z}) \to H^i(X - V, \Omega^i_{X/Z})$$

is injective, and it suffices to prove that $\Sigma_p n_p Cl_X(Z_p)$ maps to 0 in the latter group. Thus, after replacing X, W, $\tilde{W}, \ldots$ by $X - V$, $W - V$, $\tilde{W} - \psi^{-1}(V), \ldots$, we are reduced to the case when $\tilde{W}$, Z_p, Z_{pq} are all integral regular schemes, and $Z_{pq} \twoheadrightarrow Z_p$ for all p, q.

Now

$$(f)_{\tilde{W}} = \sum_{p,q} m_{pq} [Z_{pq}],$$

where if $d_{pq} = \deg(Z_{pq} \to Z_p)$, then

$$n_p = \sum_q m_{pq} d_{pq}$$

for all p (see [F], Appendix A). From Theorem 1 (b), we have commutative diagrams for each p, q

$$\begin{tikzcd}
H^0(Z_{pq}, \mathcal{O}_{Z_{pq}}) \arrow[r] \arrow[d, "(\psi|_{Z_{pq}})_*"'] & H^1(\tilde{W}, \Omega^1_{\tilde{W}/\mathbf{Z}}) \arrow[d, "\psi_*"] \\
H^0(Z_p, \mathcal{O}_{Z_p}) \arrow[r] & H^i(X, \Omega^i_{X/\mathbf{Z}})
\end{tikzcd}$$

From Theorem 1 (f), $1 \in H^0(Z_{pq}, \mathcal{O}_{Z_{pq}})$ maps to $d_{pq} \in H^0(Z_p, \mathcal{O}_{Z_p})$. Hence if

$$\alpha = \sum_{p,q} Cl_{\tilde{W}}(Z_{pq}) \in H^1(\tilde{W}, \Omega^i_{\tilde{W}/\mathbf{Z}}),$$

then

$$\psi_*(\alpha) = \sum_p n_p Cl_X(Z_p) \in H^i(X, \Omega^i_{X/\mathbf{Z}}).$$

Thus it suffices to prove that $\alpha = 0$. But this is immediate from Theorem 1 (g), since $\Sigma m_{pq}[Z_{pq}] = 0 \in H^1(\tilde{W}, \mathcal{O}^*_{\tilde{W}})$. □

Remarks.

1. By construction, if $Z \subset X$ is a closed integral subscheme of codimension i in X with $X \cap S = \phi$, then

$$Cl_X(Z) \in H^i(X, \Omega^i_{X/\mathbf{Z}})$$

is the image of a "localised" cycle class

$$\tilde{C}l_X(Z) \in H^i_Z(X, \Omega^i_{X/\mathbf{Z}}).$$

In the generality that we consider above, we do not know if there is an intersection product making $\oplus_i CH^i(X, S)$ into a ring. However, let Z_1, Z_2 be two integral closed subschemes of codimension i, j respectively, with $(Z_1 \cup Z_2) \cap S = \phi$. If $T_i \subset Z_i$ is the singular locus, assume that $Z_1 \cap Z_2$ is irreducible of codimension $i+j$, and is generically reduced; in particular, $Z_1 \cap Z_2$ is not contained in $T_1 \cup T_2$. Let $Z = (Z_1 \cap Z_2)_{\mathrm{red}}$. Then one can show that

$$\tilde{C}l_X(Z_1) \cup \tilde{C}l_X(Z_2) = \tilde{C}l_X(Z) \in H^{i+j}_Z(X, \Omega^{i+j}_{X/\mathbf{Z}}).$$

In certain contexts, it suffices to consider such intersections (see [F], Example 11.4.2); thus for X smooth and quasi-projective over an algebraically closed field,

$$Cl_X : \oplus CH^i(X) \to \oplus H^i(X, \Omega^i_{X/\mathbf{Z}})$$

is a ring homomorphism.

2. If X is smooth over a field k, then the composite

$$CH^i(X) \to H^i(X, \Omega^i_{X/\mathbf{Z}}) \to H^i(X, \Omega^i_{X/k})$$

is the standard cycle class (see [EZ]).

3. Applications

In this section, we give two applications of the existence of cycle classes in Hodge cohomology. The first one uses the standard cycle class, constructed in [EZ] (see Remark 2 in Section 2).

Application 1. Let X be a smooth, projective variety over a field k, and assume that X has a "cell decomposition" i.e., there is a filtration of X by closed subsets

$$\phi = X_0 \subset X_1 \subset \cdots \subset X_m = X$$

such that $X_i - X_{i-1} \cong \mathbf{A}_k^{n_i}$, an affine space of dimension n_i, for some integers n_i. Then we will show that:

(i) $H^i(X, \Omega^j_{X/k}) = 0 \, \forall i \neq j$;
(ii) the cycle map $CH^i(X) \otimes_{\mathbf{Z}} k \to H^i(X, \Omega^i_{X/k})$ is surjective for each i.

Both of these are standard consequences of Hodge theory, if $k = \mathbf{C}$; we give a proof valid in any characteristic. Though the argument given below is standard (see [Kl] where the case $k = \mathbf{C}$ is treated), the above facts do not seem to have been noted in the literature in the case char. $k = p > 0$; the special case when $X = G/P$, where G is a semisimple algebraic group, P a parabolic subgroup, has been dealt with by representation theoretic methods (see [Ma]).

To prove (i) and (ii), note that for any variety with a cell decomposition, $CH^i(X)$ is generated by the classes of all the $\overline{X_j - X_{j-1}}$ such that $\dim X - n_j = i$ (where $X_j - X_{j-1} \cong \mathbf{A}^{n_j}$). If X has a cell decomposition, so does $X \times X$, where the "cells" are of the form

$$\mathbf{A}^{n_j} \times \mathbf{A}^{n_l} \cong X_j \times X_l - ((X_{j-1} \times X_l) \cup (X_j \times X_{l-1})).$$

In particular, $CH^m(X \times X)$ is generated as an abelian group by cycles of the form $Y \times Z$, where $Y, Z \subset X$ are irreducible, and the sum of their codimensions is m.

If $\operatorname{codim}_X Y = r$, $\operatorname{codim}_X Z = s$, then we have cycle classes

$$Cl_X(Y) \in H^r(X, \Omega^r_{X/k}),$$
$$Cl_X(Z) \in H^s(X, \Omega^s_{X/k}),$$
$$Cl_{X \times X}(Y \times Z) \in H^{r+s}(X \times X, \Omega^{r+s}_{X \times X/k}).$$

Then $Cl_X(Y) \otimes Cl_X(Z) \mapsto Cl_{X \times X}(Y \times Z)$ under the external product

$$H^r(X, \Omega^r_{X/k}) \otimes H^s(X, \Omega^s_{X/k}) \to H^{r+s}(X \times X, p_1^* \Omega^r_{X/k} \otimes p_2^* \Omega^s_{X/k})$$
$$\to H^{r+s}(X \times X, \Omega^{r+s}_{X \times X/k})$$

where $p_i: X \times X \to X$, $i = 1, 2$, are the projections. The isomorphism

$$\Omega^m_{X \times X/k} \cong \oplus_{t+u=m} p_1^* \Omega^t_{X/k} \otimes p_2^* \Omega^u_{X/k}$$

yields the Kunneth decomposition

$$H^m(X \times X, \Omega^m_{X \times X/k}) \cong \bigoplus_{\substack{r+s=m \\ t+u=m}} H^r(X, \Omega^t_{X/k}) \otimes H^s(X, \Omega^u_{X/k})$$

which is obtained from the various external products

$$H^r(X,\Omega^t_{X/k})\otimes H^s(X,\Omega^u_{X/k})\to H^m(X\times X,\Omega^m_{X\times X/k}).$$

Thus only one Kunneth component of $Cl_{X\times X}(Y\times Z)$ can be non-zero, namely $Cl_X(Y)\otimes Cl_X(Z)$. Hence the image of the cycle class map

$$Cl_{X\times X}:CH^m(X\times X)\to H^m(X\times X,\Omega^m_{X\times X/k})$$

is contained in the direct sum of the Kunneth components

$$\oplus_{r+s=m}H^r(X,\Omega^r_{X/k})\otimes H^s(X,\Omega^s_{X/k}).$$

Apply this in particular to the diagonal $\Delta_X\subset X\times X$; we see that if $n=\dim X$, then

$$Cl_{X\times X}(\Delta_X)\in\oplus_{r+s=n}H^r(X,\Omega^r_{X/k})\otimes H^s(X,\Omega^s_{X/k}).$$

But if we identify (by Serre duality)

$$H^r(X,\Omega^t_{X/k})\cong H^{n-r}(X,\Omega^{n-t}_{X/k})^*$$

for all r,t, then if $r+s=t+u=n$, we have

$$H^r(X,\Omega^t_{X/k})\otimes H^s(X,\Omega^u_{X/k})\cong \mathrm{End}_k(H^s(X,\Omega^u_{X/k})),$$

and the (r,t,s,u) Kunneth component of $Cl_{X\times X}(\Delta_X)$ corresponds to the identity endomorphism of $H^s(X,\Omega^u_{X/k})$. We deduce that if $s\neq u$, the identity endomorphism of $H^s(X,\Omega^u_{X/k})$ vanishes, i.e., $H^s(X,\Omega^u_{X/k})=0$ for $s\neq u$. This proves (i).

Next, if Y and Z are as above, and $r+s=n$, the endomorphism of $H^s(X,\Omega^s_{X/k})$ induced by

$$Cl_X(Y)\otimes Cl_X(Z)\in H^r(X,\Omega^r_{X/k})\otimes H^s(X,\Omega^s_{X/k})\cong \mathrm{End}(H^s(X,\Omega^s_{X/k}))$$

is the composite

$$H^s(X,\Omega^s_{X/k})\xrightarrow{\cup Cl_X(Y)}H^n(X,\Omega^n_{X/k})\xrightarrow{\mathrm{tr}}k\xrightarrow{\cdot Cl_X(Z)}H^s(X,\Omega^s_{X/k}),$$

i.e., has image contained in the subspace $k\cdot Cl_X(Z)$. Since the (r,r,s,s) Kunneth component of $Cl_{X\times X}(\Delta_X)$ is a $\mathbf{Z}$-linear combination of such classes, the image of the identity endomorphism of $H^s(X,\Omega^s_{X/k})$ is contained in the image of

$$Cl_X:CH^s(X)\otimes_{\mathbf{Z}}k\to H^s(X,\Omega^s_{X/k})$$

i.e., this map is surjective. This proves (ii).

Note that if char. $k=p>0$, and X lifts to a smooth projective variety $\tilde{X}/\tilde{k}$ (where $\tilde{k}$ is the quotient field of a discrete valuation ring of characteristic 0 with residue field k), such that the specialisation maps

$$CH^i(\tilde{X})\to CH^i(X)$$

are isomorphisms of torsion free abelian groups, then

$$CH^i(\tilde{X})\otimes_{\mathbf{Z}}\tilde{k}\cong H^i(\tilde{X},\Omega^i_{\tilde{X}/\tilde{k}})$$
$$\Rightarrow CH^i(X)\otimes_{\mathbf{Z}}k\cong H^i(X,\Omega^i_{X/k}),$$

since in any case

$$\dim_k H^i(X, \Omega^i_{X/k}) \geqslant \dim_{\tilde{k}} H^i(\tilde{X}, \Omega^i_{\tilde{X}/\tilde{k}})$$

by the semicontinuity theorem. This is the case for the homogeneous spaces $X = G/P$, where G is a semisimple algebraic group, and P is a reduced parabolic subgroup.

Application 2. We give a 'natural' proof for the infinite dimensionality theorem for the Chow group of zero cycles $CH^n(X,S)$ on a normal projective variety $X/\mathbf{C}$ (of dimension n) with $H^n(X, \mathcal{O}_X) \neq 0$.

In fact, let X be an irreducible projective variety with $H^n(X, \mathcal{O}_X) \neq 0$, and $Y \subset X$ a closed subscheme. We will show that the zero cycles supported on Y do not generate $CH^n(X,S)$; this implies the infinite dimensionality theorem for normal projective varieties (see [S], [S2]).

Suppose that X, Y are defined over a countable algebraically closed subfield $k \subset \mathbf{C}$. Let X_0, Y_0, S_0 be the models over k. There is an embedding of the function field $k(X_0)$ as a k-subalgebra of $\mathbf{C}$. Let K be the algebraic closure of its image in $\mathbf{C}$; we thus have morphisms

$$\operatorname{Spec} \mathbf{C} \to \operatorname{Spec} K \to \operatorname{Spec} k(X_0) \to X_0,$$

where the last map is the inclusion of the generic point. Let $\eta \in X$ be the corresponding closed point. We claim that $[\eta] \in CH^n(X,S)$ is not contained in the subgroup of $CH^n(X,S)$ generated by the classes of points of $Y - S$.

Let

$$\theta_1 : \Omega^1_{X/\mathbf{Z}} \to \Omega^1_{X/X_0} \cong \mathcal{O}_X \otimes_{\mathbf{C}} \Omega^1_{\mathbf{C}/k}$$

be the natural map. This induces a natural map for each $i \leqslant n$

$$\theta_i = \overset{i}{\wedge}\, \theta_1 : \Omega^i_{X/\mathbf{Z}} \to \Omega^i_{X/X_0} \cong \mathcal{O}_X \otimes_{\mathbf{C}} \Omega^i_{\mathbf{C}/k},$$

and hence a map

$$Cl' : CH^i(X,S) \to H^i(X, \mathcal{O}_X) \otimes_{\mathbf{C}} \Omega^i_{\mathbf{C}/k}.$$

We claim that

(i) If $P \in Y - S$ is a closed point, then $Cl'(P) = 0$.
(ii) $Cl'(\eta) \neq 0$.

This will prove that $[\eta]$ is not contained in the subgroup generated by points of $Y - S$.

Let $P \in X - S$ be a closed point. Let $L \subset \mathbf{C}$ be a k-subalgebra of $\mathbf{C}$ which has finite transcendence degree over k, such that P corresponds to a closed L-rational point. Let $X_L = X_0 \times_{\operatorname{Spec} k} \operatorname{Spec} L$, etc. and let $Q \in X_L$ be the L-rational point corresponding to P. We can define a map

$$Cl'_L : CH^n(X_L, S_L) \to H^n(X_L, \mathcal{O}_{X_L}) \otimes_L \Omega^n_{L/k}$$

which is analogous to Cl'. Then by Theorem 1(c), we see that $Cl'(P)$ is the image of

$Cl'_L(Q)$ under the natural map

$$H^n(X_L, \mathcal{O}_{X_L}) \otimes_L \Omega^n_{L/k} \to H^n(X, \mathcal{O}_X) \otimes_{\mathbf{C}} \Omega^n_{\mathbf{C}/k}.$$

Since

$$H^n(X, \mathcal{O}_X) \otimes_{\mathbf{C}} \Omega^n_{\mathbf{C}/k} \cong H^n(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^n_{\mathbf{C}/k},$$

and we have a similar expression for $H^n(X_L, \mathcal{O}_{X_L}) \otimes_L \Omega^n_{L/k}$, and $\Omega^n_{L/k} \subset \Omega^n_{\mathbf{C}/k}$, we see that

$$Cl'(P) = 0 \Leftrightarrow Cl'_L(Q) = 0.$$

Let $\mathcal{I}_Q$ be the ideal sheaf of Q on X_L. There is an exact sequence

$$\mathcal{I}_Q/\mathcal{I}_Q^2 \xrightarrow{\psi} \Omega^1_{X_L/X_0} \otimes \mathcal{O}_Q \xrightarrow{\chi} \Omega^1_{Q/X_0} \to 0$$

where Q is regarded as an X_0 scheme through the morphism $X_L \to X_0$. This defines an element

$$\overset{n}{\wedge} \psi \in \mathrm{Hom}(\overset{n}{\wedge} (\mathcal{I}_Q/\mathcal{I}_Q^2), \Omega^n_{X_L/X_0} \otimes \mathcal{O}_Q)$$
$$\cong \mathrm{Ext}^n(\mathcal{O}_Q, \Omega^n_{X_L/X_0}).$$

From the definition of Cl'_L, we see that $Cl'_L(Q)$ is the image of this element $\overset{n}{\wedge} \psi$ under the natural map

$$\mathrm{Ext}^n(\mathcal{O}_Q, \Omega^n_{X_L/X_0}) \to \mathrm{Ext}^n(\mathcal{O}_{X_L}, \Omega^n_{X_L/X_0})$$
$$= H^n(X_L, \Omega^n_{X_L/X_0}) \cong H^n(X_L, \mathcal{O}_{X_L}) \otimes_L \Omega^n_{L/k}.$$

We now let P be a point of $Y - S$. We then claim that if L, Q are chosen as above, then $Cl'_L(Q) = 0$. It suffices to prove that $\overset{n}{\wedge} \psi = 0$. Now $\mathcal{I}_Q/\mathcal{I}_Q^2$ is a skyscraper sheaf supported at Q whose stalk is an L-vector space of dimension n; hence $\overset{n}{\wedge} \psi \neq 0 \Leftrightarrow \psi$ is *injective.*

The other two terms in the above exact sheaf sequence are also skyscraper sheaves with stalks which are finite dimensional L-vector spaces. This is clear for the last term; to see it for the middle term, use the isomorphism

$$\Omega^1_{X_L/X_0} \otimes \mathcal{O}_Q \cong \Omega^1_{L/k} \otimes \mathcal{O}_Q;$$

the dimension of the middle term thus equals the transcendence degree of L/k.

Since $Q \in Y_L$, the morphism $Q \to X_0$ factors through $Y_0 \subset X_0$. Hence if $T \subset X_0$ is the Zariski closure of the image of Q in X_0, then $\Omega^1_{Q/X_0} = \Omega^1_{Q/T}$ has dimension equal to the transcendence degree of L over its subfield $k(T)$. Thus the kernel of χ has dimension equal to the transcendence degree of $k(T)$ over k. But this is $< n$; hence ψ is not injective, and $Cl'_L(Q) = 0$.

To prove (ii), take $L = K$, and Q to be the closed point of X_K obtained from the natural map $\operatorname{Spec} K \to X_0$. It suffices to prove that $Cl'_K(Q) \neq 0$. Now ψ is an isomorphism. Thus

$$\mathrm{Hom}(\overset{n}{\wedge} (\mathcal{I}_Q/\mathcal{I}_Q^2), \Omega^n_{X_K/X_0} \otimes \mathcal{O}_Q) \cong \mathrm{Ext}^n(\mathcal{O}_Q, \mathcal{O}_{X_K} \otimes_K \Omega^n_{K/k})$$

is a 1-dimensional K-vector space spanned by $\overset{n}{\wedge}\psi$, and $\Omega^n_{K/k}$ is a 1-dimensional vector space.

Hence it suffices to prove the natural map

$$\mathrm{Ext}^n(\mathcal{O}_Q,\mathcal{O}_{X_K})\to H^n(X_K,\mathcal{O}_{X_K})$$

is injective, if $H^n(X_K,\mathcal{O}_{X_K})\neq 0$. By Serre duality (see [H], Ch. III, Sec. 7) there is a coherent $\mathcal{O}_{X_K}$ module $\omega^0_{X_K}$, the dualising sheaf, such that $H^0(X_K,\omega^0_{X_K})$ is dual to $H^n(X_K,\mathcal{O}_{X_K})$; in particular, $\omega^0_{X_K}$ has non-zero global sections. A section s of $\omega^0_{X_K}$ gives rise to a diagram

$$\begin{tikzcd}
\mathrm{Ext}^n(\mathcal{O}_Q,\mathcal{O}_{X_K}) \arrow[r, "s_*"] \arrow[d] \arrow[dr, phantom, "(*)"] & \mathrm{Ext}^n(\mathcal{O}_Q,\omega^0_{X_K}) \arrow[d] \\
H^n(X_K,\mathcal{O}_{X_K}) \arrow[r, "s_*"] & H^n(X_K,\omega^0_{X_K})
\end{tikzcd}$$

where the top arrow is a K-linear map between 1-dimensional K-vector spaces, which vanishes precisely when s vanishes at Q (this follows from the fact that $\omega^0_{X_K}$ is an invertible sheaf in a neighbourhood of Q, so that the Ext is isomorphic to the global sections of the corresponding sheaf $\mathcal{E}xt$).

From the lemma below, the right hand vertical arrow in the square $(*)$ is injective. Thus, the left hand vertical map is injective, provided there exists a section s which does not vanish at Q. But

$$H^0(X_K,\omega^0_{X_K})=H^0(X_0,\omega^0_{X_K})\otimes_k K$$

and for any nonzero section $s\in H^0(X_0,\omega^0_{X_0})$, $s\otimes 1$ will not vanish at Q, since Q maps to the generic point of X_0.

So we are reduced to proving the following lemma.

Lemma 11. *Let X be an irreducible projective variety of dimension n over a field K, and $Q\in X$ a smooth K-rational point. Then the natural map* $\mathrm{Ext}^n(\mathcal{O}_Q,\omega^0_X)\to H^n(X,\omega^0_X)$ *is injective.*

Proof. Let $\mu:\mathrm{Ext}^n(\mathcal{O}_Q,\omega^0_X)\to K$ be the map obtained by composing it with the trace map

$$H^n(X,\omega^0_X)\to K;$$

we claim μ is an isomorphism. If we fix a K-embedding $X\subset\mathbf{P}^N_K$, and identify ω^0_X with $\mathcal{E}xt^{N-n}_{\mathbf{P}^N}(\mathcal{O}_X,\omega_{\mathbf{P}^N})$, the trace map is the natural composite map

$$H^n(\mathbf{P}^N,\mathcal{E}xt^{N-n}_{\mathbf{P}^N}(\mathcal{O}_X,\omega_{\mathbf{P}^N}))\to\mathrm{Ext}^N_{\mathbf{P}^N}(\mathcal{O}_X,\omega_{\mathbf{P}^N})\to H^N(\mathbf{P}^N,\omega_{\mathbf{P}^N})=K.$$

The map μ factors through the natural map $H^n_Q(X,\omega^0_X)\to H^n(X,\omega^0_X)$, and we have a commutative diagram

$$\begin{tikzcd}
H^n_Q(X,\omega^0_X) \arrow[r] \arrow[d] & H^N_Q(\mathbf{P}^N,\omega_{\mathbf{P}^N}) \arrow[d] \\
H^n(X,\omega^0_X) \arrow[r] & H^N(\mathbf{P}^N,\omega_{\mathbf{P}^N})
\end{tikzcd}$$

If U is the open subscheme of smooth points of X, then there is an isomorphism

$$\mathrm{Ext}^n_X(\mathcal{O}_Q, \omega^0_X) \cong \mathrm{Ext}^n_U(\mathcal{O}_Q, \omega_U)$$

where $\omega_U \cong \Omega^n_{U/K}$ We have a composite Gysin map

$$H^0(Q, \mathcal{O}_Q) \to \mathrm{Ext}^n_U(\mathcal{O}_Q, \omega_U) \to H^n_Q(U, \omega_U)$$

induced by the Gysin map of Theorem 1 for the inclusion $Q \to U$. The image of $1 \in H^0(Q, \mathcal{O}_Q)$ in $\mathrm{Ext}^n(\mathcal{O}_U, \omega_U)$ is a generator of this 1-dimensional vector space.

If $S = X - U$, there are Gysin maps for each of the maps $Q \to U$, $U \to \mathbf{P}^N - S$, $Q \to \mathbf{P}^N - S$. From Theorem 1 (b), we are thus reduced to showing that the following composite is the identity:

$$H^0(Q, \mathcal{O}_Q) \to H^N_Q(\mathbf{P}^N, \omega_{\mathbf{P}^N}) \to H^N(\mathbf{P}^N, \omega_{\mathbf{P}^N}) \xrightarrow{tr} K$$

This is a particular case of lemma 8. □

Our results can also be used to discuss examples of M V Nori (unpublished) of indecomposable projective modules of rank n over n-dimensional affine algebras over $\mathbf{C}$. Let X be a smooth projective variety over $\mathbf{C}$, and let $Y \subset X$ be an effective divisor, $U = X - Y$ the complement. Let $k \subset \mathbf{C}$ be a countable algebraically closed field of definition for X, Y, U with corresponding models X_0, Y_0, U_0 and $K \subset \mathbf{C}$ an embedding of the algebraic closure of the function field $k(X_0)$ as a k-subalgebra of $\mathbf{C}$. Let $\eta \in X$ be the closed point determined by the generic point of X_0, and the embedding $K \subset \mathbf{C}$; then in fact $\eta \in U$.

We claim that if $Z \subset Y$ is a subvariety which has codimension i in X, then

$$Cl'(Z) = 0 \text{ in } H^i(X, \mathcal{O}_X) \otimes_{\mathbf{C}} \Omega^i_{\mathbf{C}/k}.$$

This implies that the map

$$Cl' : CH^i(X) \to H^i(X, \mathcal{O}_X) \otimes_{\mathbf{C}} \Omega^i_{\mathbf{C}/k}$$

factors through the natural surjection $CH^i(X) \twoheadrightarrow CH^i(U)$, to yield a homomorphism

$$Cl' : CH^i(U) \to H^i(X, \mathcal{O}_X) \otimes_{\mathbf{C}} \Omega^i_{\mathbf{C}/k}.$$

The claim follows as in the case $Z =$ point by a simple computation with transcendence degrees.

In particular, suppose that $H^i(X, \mathcal{O}_X) = 0$ for $i < n$. Then the composite product maps (see Remark 1 of the previous section)

$$CH^i(X) \otimes CH^{n-i}(X) \to CH^n(X) \xrightarrow{Cl'} H^n(X, \mathcal{O}_X) \otimes_{\mathbf{C}} \Omega^n_{\mathbf{C}/k}$$

are zero, for $0 < i < n$. Since any zero cycle supported on Y also has zero image under Cl', the map $Cl' : CH^n(H) \to H^n(X, \mathcal{O}_X) \otimes \Omega^n_{\mathbf{C}/k}$ factors through the natural (surjective) map $CH^n(X) \twoheadrightarrow CH^n(U)$. Thus, we conclude that $[\eta] \in CH^n(U)$ is not in the image of any of the product maps

$$CH^i(U) \otimes CH^{n-i}(U) \to CH^n(U).$$

Now suppose $U = \operatorname{Spec} A$ is affine, and choose a free resolution over A of the residue field at η,

$$F_{n-1} \xrightarrow{f} F_{n-2} \to \cdots F_1 \to F_0 \to \mathbf{C}(\eta) \to 0.$$

Then $\ker f = P'$ is a projective A-module with

$$[P'] - [A^{\oplus r}] = [\mathbf{C}(\eta)] \in K_0(A)$$

where $r = \operatorname{rank} P'$. By Bass' stability theorem, we can then find a projective A-module P of rank n with

$$[P] - [A^{\oplus n}] = [\mathbf{C}(\eta)] \in K_0(A).$$

Then the nth Chern class of P is given by

$$c_n(P) = (-1)^{n-1}(n-1)!\,[\eta] \in CH^n(A) = CH^n(U).$$

If P is decomposable, an easy computation shows that $c_n(P)$ would lie in the subgroup of $CH^n(U)$ generated by the images of the products

$$CH^i(U) \otimes CH^{n-i}(U) \to CH^n(U).$$

for all $0 < i < n$, which is not the case, since $(-1)^{n-1}(n-1)!\,Cl'([\eta]) \neq 0$. Hence P is indecomposable.

In fact, Levine has constructed a theory of Chern classes for vector bundles with values in $CH^*(X, S)$ for quasi-projective varieties with isolated singularities, where S is the singular locus (he has a theory for more general singularities, but which uses cycles meeting the singular locus 'nicely', for which we do not have the cycle class). This allows us to extend Nori's examples to the case when X is a projective variety with isolated singularities with $H^n(X, \mathcal{O}_X) \neq 0$, $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$, and Y is a hyperplane section disjoint from S.

More generally, let X be a projective variety over $\mathbf{C}$ with isolated singularities, Y an effective divisor disjoint from the singular locus S of X, and $U = X - Y$; let X, Y, U are defined over k, an algebraically closed subfield of $\mathbf{C}$. Then there is a well defined ring homomorphism

$$CH^*(U, S) \to \oplus H^i(X, \mathcal{O}_X) \otimes_{\mathbf{C}} \Omega^i_{\mathbf{C}/k},$$

where the Chow ring is that defined by Levine.

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Some remarks on subgroups determined by certain ideals in integral group rings

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Abstract. Let G be a group, ZG the integral group ring of G and $I(G)$ its augmentation ideal. Subgroups determined by certain ideals of ZG contained in $I(G)$ are identified. For example, when $G = HK$, where H, K are normal subgroups of G and $H \cap K \subseteq \zeta(H)$, then the subgroups of G determined by $I(G)I(H)I(G)$, and $I^3(G)I(H)$ are obtained. The subgroups of any group G with normal subgroup H determined by

(i) $I^2(G)I(H) + I(G)I(H)I(G) + I(H)I^2(G)$, when $H' \subseteq [H, G, G]$ and
(ii) $I(G)I(H)I(G)$ when $\deg H^2(G/H', T) \leqslant 1$,

are computed. The subgroup of G determined by $I^n(G) + I(G)I(H)$ when H is a normal subgroup of G with G/H free Abelian is also obtained

Keywords. Group ring; augmentation ideal; dimension property; generalized extra special p-group.

1. Introduction

Let G be a group, ZG the integral group ring of G and $I(G)$ its augmentation ideal. If G is a free group and H a normal subgroup of G, then the subgroups of G determined by the ideals $I^2(G)I(H)$, $I(H)I(G)I(H)$, $I(G)I(H)I(G)$, $I(G)I(G')I(G)$, $I^2(G)I^2(H)$, $I(H)I^2(G)I(H)$ and $I(G)I^n(H)I(G)$ of ZG have been investigated by several authors (cf. [1], chapter V, §4). Khambadkone ([6], [7]) investigated the structure of the subgroups $G \cap (1 + I^2(G)I(H))$ and $G \cap (1 + I^3(G)I(H))$ of G when G is finite and is semidirect product $H \rtimes K$ of a normal subgroup H and a subgroup K. Ram Karan and Vermani [5] computed the subgroup $G \cap (1 + I^2(G)I(H))$ of G when G is any group with normal subgroups H, K such that $G = HK$ and $H \cap K \subseteq \zeta(H)$, the centre of H. Continuing these investigations for such a group G, we obtain the subgroups of G determined by (a) $I(G)I(H)I(G)$ and (b) $I^3(G)I(H)$, $I^2(G)I(H)I(G)$, $I(G)I(H)I^2(G)$ when $\deg H^2(H/H_3(H \cap K), T) \leqslant 2$, where T is the additive group of rationals mod 1 regarded as a trivial G-module.

There do exist groups with the above property. For example, if G is a generalized extra special p-group and H a normal subgroup of G, then there exists a normal subgroup K of G such that $G = HK$ and $H \cap K \subseteq \zeta(H)$.

If H is a normal subgroup of any group G, we prove

a) $G \cap (1 + I(G)I^2(H) + I(H)I(G)I(H) + I^2(H)I(G)) = [G, H, H]$;

b) $G\cap(1+I^2(G)I(H)+I(G)I(H)I(G)+I(H)I^2(G))=[H,G,G]$, provided $H'\subseteq[H,G,G]$; and

c) $G\cap(1+I(G)I(H)I(G))=[H',G]$, when $\deg H^2(G/H',T)\leqslant 1$.

Passi (cf [8], Theorem V. 5.4) proved that $G\cap(1+I^n(G)+I(G)I(H))=G_n$, when H is a normal Abelian subgroup of G with G/H cyclic. We generalize this result of Passi and prove that $G\cap(1+I^n(G)+I(G)I(H))=H'G_n$, when H is a normal subgroup of G with G/H free Abelian.

Notation. For a group G, we write G_n for the nth term of the lower central series of G with $G_1=G$. We also write G' for G_2. Throughout, T stands for the additive group of rationals mod 1 regarded as a trivial G-module. For a normal subgroup H of a Group G, we write $\mathscr{H}(n)$ for $\Sigma_{i+j+1=n}I^i(G)I(H)I^j(G)$.

2. The case of $G=HK$ with $H\cap K\subseteq\zeta(G)$

Let G be a group, H, K the subgroups of G with K normal in G and $G=HK$. Let S be a left transversal of $H\cap K$ in K. Then S is also a left transversal of H in G. Let θ^*: $ZG\to ZH$ be the map induced by the map θ: $G\to H$, $\theta(sh)=h$, $s\in S$, $h\in H$ and linearity.

PROPOSITION 2.1

If $[H\cap K,H]\subseteq H_3$, *then*

$$G\cap(1+I(H)I(G)I(H))=H_3.$$

Proof. Since $G=HK$ and K is normal in G,

$$I(H)I(G)I(H)\subseteq I^3(H)+I(K)I(H).$$

Therefore $G\cap(1+I(H)I(G)I(H))\subseteq H\cap(1+I^3(H)+I(K)I(H))$.
An application of the homomorphism θ^* shows that

$$\begin{aligned}H\cap(1+I^3(H)+I(K)I(H))&=H\cap(1+I^3(H)+I(H\cap K)I(H))\\&=H_3.\ \text{(by [8], Theorem V.5.9)}.\end{aligned}$$

Reverse inclusion being trivial, the result follows.

Lemma 2.2 *If both* H, K *are normal in* G *and* $H\cap K\subseteq\zeta(H)$, *then*

i) $I^n(G)I(H)\subseteq I^{n+1}(H)+I(K)I(H)$;

ii) $I(H)I^n(G)\subseteq I^{n+1}(H)+I(K)I(H)+I(\underbrace{[H,K,\dots,K]}_{n\text{-times}})$

iii) $I^n(G)I(H)I^m(G)\subseteq I^{m+n+1}(H)+I(K)I(H)$, *provided* $n\geqslant 1$;

iv) $I(H)I^n(G)\subseteq I^{n+1}(H)+I(K)I(H)$, *provided* $H\cap K\subseteq\zeta(G)$.

Theorem 2.3 *With H, K as in lemma 2.2,*

$$G\cap(1+I(G)I(H)I(G))=H_3, \text{ and}$$
$$G\cap(1+I^2(G)I(H)+I(H)I^2(G))=H_3, \textit{ provided } H\cap K\subseteq\zeta(G).$$

Proof. Let $G\cap(1+I(G)I(H)I(G))=X$ and $Y=G\cap(1+I^2(G)I(H)+I(H)I^2(G))$. Observe that $H_3\subseteq X\cap Y$. Both X and Y are contained in $G\cap(1+ZGI(H))=H$. Therefore, by lemma 2.2, X and Y are contained in $H\cap(1+I^3(H)+I(K)I(H))$. An application of the map θ^*: $ZG\rightarrow ZH$ shows that $H\cap(1+I^3(H)+I(K)I(H))=H\cap(1+I^3(H)+I(H\cap K)I(H))$ which equals H_3 (by [8], Theorem V.5.9).

Proceeding as in the above case and using Theorem V.5.1 of [8] we can also obtain

PROPOSITION 2.4

If H, K are as in lemma 2.2. and deg $H^2(H/H_3(H\cap K), T)\leqslant 2$, *then*

$$\begin{aligned}G\cap(1+I^3(G)I(H))&=G\cap(1+I(H)I^3(G))=G\cap(1+I^2(G)I(H)I(G))\\&=G\cap(1+I(G)I(H)I^2(G))=D_4(H).\end{aligned}$$

In case $H\cap K\subseteq\zeta(G)$, we also have

$$G\cap(1+I^3(G)I(H)+I(H)I^3(G))=D_4(H).$$

For the existence of normal subgroups H, K of a group G with $G=HK$ and $H\cap K\subseteq\zeta(H)$ or $\zeta(G)$, we have

PROPOSITION 2.5

Let G be a finite p-group and H be a normal subgroup of G with nilpotency class 2 such that $H\cap\Phi(G)\leqslant\zeta(H)$, where $\Phi(G)$ is the Frattini subgroup of G. Then there exists a normal subgroup K of G such that $G=HK$, $H\cap K\subseteq\zeta(H)$. If, however, H is Abelian and $H\cap\Phi(G)\subseteq\zeta(G)$, then $H\cap K\subseteq\zeta(G)$.

Proof. Since G is a finite p-group,

$$\Phi(G/\zeta(H))=G'G^p\zeta(H)/\zeta(H)=\Phi(G)\zeta(H)/\zeta(H) \text{ and}$$
$$H/\zeta(H)\cap\Phi(G/\zeta(H))=(H\cap\Phi(G))\zeta(H)/\zeta(H)=1.$$

A result of Gaschütz states that if A is an Abelian normal subgroup of a finite group G with $A\cap\Phi(G)=1$, then A has a complement in G (a proof of this follows on the lines of the proof of Theorem 7.4.14, p. 169 of [11]). Therefore, there exists a normal subgroup $K/\zeta(H)$ of $G/\zeta(H)$ such that

$$G/\zeta(H)=(K/\zeta(H))(H/\zeta(H)) \text{ and}$$
$$(H/\zeta(H))\cap(K/\zeta(H))=1.$$

Then K is a normal subgroup of G with $G=HK$ and $H\cap K=\zeta(H)$. The other assertion follows on similar lines.

Let G be a generalized extra special p-group, p a prime i.e. G is a finite p-group with $\zeta(G)$ cyclic and G' of order p. Then $G' \subseteq \zeta(G)$ and $G/\zeta(G)$ is elementary Abelian. Therefore, G is nilpotent of class 2 and $G^p \subseteq \zeta(G)$. Also

$$H \cap \Phi(G) = H \cap G'G^p \subseteq H \cap \zeta(G) \subseteq \zeta(H).$$

There then exists a normal subgroup K of G such that $G = HK$ and $H \cap K \subseteq \zeta(H)$. If $p \neq 2$ or G/G' is finite 2-group of rank $\leqslant 3$, then we also have ([8, Theorem V.6.8] and [12]) deg $H^2(H/H \cap K, T) \leqslant 2$. We thus have

PROPOSITION 2.6

If G is a generalized extra special p-group and H is a normal subgroup of G, then

a) $G \cap (1 + I(H)I(G)I(H)) = H_3$; *and*
b) $G \cap (1 + I^3(G)I(H)) = G \cap (1 + I^2(G)I(H)I(G)) = D_4(H)$, *provided* $p \neq 2$ *or* G/G' *is of rank* $\leqslant 3$.

3. The subgroups determined by $I(G)I(H)I(G)$ and $\mathscr{H}(3)$

PROPOSITION 3.1

For any normal subgroup H of a group G, $G \cap (1 + I(G)I^2(H) + I(H)I(G)I(H) + I^2(H)I(G)) = [G, H, H]$.

Proof. It is easy to see that

$$[G, H, H] \subseteq G(1 + I(G)I^2(H) + I(H)I(G)I(H) + I^2(H)I(G)) = X \text{ (say)}.$$

Factoring G by $[G, H, H]$, if necessary, we may assume that $[G, H] \subseteq \zeta(H)$. Then

$$\begin{aligned} X &\subseteq G \cap (1 + I(G)I^2(H) + I([H, G])I(H)) \\ &= H \cap (1 + I^3(H) + I([H, G])I(H)) \\ &= H_3 = 1 \text{ (by [8], Theorem V.5.9)} \end{aligned}$$

PROPOSITION 3.2

Let G be any group and H a subgroup of G such that deg $H^2(G/K', T) \leqslant 1$, where $K = H^G$ is the normal subgroup of G generated by H. Then

$$G \cap (1 + I(G)I(H)I(G)) = [K', G].$$

Proof. Observe that $I(G)I(H)I(G) = I(G)I(K)I(G)$. We may, therefore, assume that H itself is normal in G. It is clear that

$$[H', G] \subseteq G \cap (1 + I(G)I(H)I(G)) \subseteq G \cap (1 + I(G)I(H)) = H'.$$

By factoring with $[H', G]$, if necessary, we may assume that $H' \subseteq \zeta(G)$.

Let $1 \neq x \in G \cap (1 + I(G)I(H)I(G))$ and $\alpha : H' \to T$ be a homomorphism such that $\alpha(x) \neq 0$. It follows from Theorem V.5.1. of Passi [8] that α can be extended to a polynomial map $\beta : G \to T$ of degree $\leqslant 2$. In particular, the extension of β to ZG by linearity vanishes on $I(G)I(H)I(G)$. Therefore,

$$\alpha(x) = \beta(x - 1) = 0, \text{ a contradiction.}$$

Since for an extra special p-group $G, p \neq 2$, $\deg H^2(G, T) \leqslant 1$ [12], we have

COROLLARY 3.3

If G is a group with a normal subgroup H such that G/H' is an extra special p-group, $p \neq 2$, then

$$G \cap (1 + I(G)I(H)I(G)) = [H', G].$$

PROPOSITION 3.4

If H is a normal subgroup of G such that $H' \subseteq [H, G, G]$, then the subgroup $G \cap (1 + \mathscr{H}(3)) = X$ (say) of G equals $[H, G, G]$.

Proof. It is clear that $[H, G, G] \subseteq X$.

$$\text{Also, } I(G)I(H)I(G) + I(H)I^2(G) \subseteq I(H)I(G).$$

Passi has proved (cf [13]) that for any normal subgroup N of G,

$$G \cap (1 + I^n(G)I(N) + I(N)I(G)) = N'[N, \underbrace{G, G, \ldots, G}_{n\text{-times}}] \text{ for all } n \geqslant 1.$$

Therefore,

$$X \subseteq G \cap (1 + I^2(G)I(H) + I(H)I(G)) = [H, G, G]\, H' = [H, G, G].$$

4. The subgroup $G \cap (1 + I^n(G) + I(G)I(H))$

4.1

Let G be a group with a normal subgroup H such that G/H is free Abelian. Let $X = \{x_\delta \mid \delta \in \Delta\}$ be a set of elements of G such that $\{x_\delta H \mid \delta \in \Delta\}$ is a basis of G/H. We may suppose that the index set Δ is well ordered. As $G' \subseteq H$, we can choose a set S of representatives of H in G consisting of elements of the form

$$x_{\delta_1}^{t_1} x_{\delta_2}^{t_2} \ldots x_{\delta_n}^{t_n},\ t_i \in Z,\ n \geqslant 1,\ \delta_1 < \delta_2 < \ldots < \delta_n.$$

Let L_n be the Z-submodule of $I(G)$ generated by elements of the form

$$(x_{\delta_1}^{\epsilon_1} - 1)(x_{\delta_2}^{\epsilon_2} - 1) \ldots (x_{\delta_n}^{\epsilon_n} - 1),\ \epsilon_i = 1 \text{ or } -1$$

for every i and $\delta_1 \leqslant \delta_2 \leqslant \ldots \leqslant \delta_n$. For $m \geqslant 2$, let $L^{(m)} = \Sigma_{n \geqslant m} L_n$.

Theorem 4.2. *For* $n \geqslant 2$,

$$I^n(G) = I^{n-1}(G)I(H) + I^{n-2}(G)I(G') + \ldots + I(G)I(G_{n-1}) + I(G_n) \oplus L^{(n)}.$$

Proof. Let $x, y \in G$, $a, b \in H$. Since $I(H)I(G) \subseteq I(G)I(H) + I(G')$,

$$(xa-1)(yb-1) \equiv (x-1)(y-1) \text{ and}$$
$$(y-1)(x-1) \equiv (x-1)(y-1)$$

modulo $I(G)I(H) + I(G')$. Therefore,

$$I^2(G) = I(G)I(H) + I(G') + L^{(2)}.$$

Suppose that $n \geqslant 2$ and that

(4.3) $$I^n(G) = I^{n-1}(G)I(H) + I^{n-2}(G)I(G') + \ldots + I(G)I(G_{n-1}) + I(G_n) + L^{(n)}.$$

Then

$$I^{n+1}(G) \subseteq I^n(G)I(H) + I^{n-1}(G)I(G') + \ldots + I(G)I(G_n) + I(G)\,L^{(n)}.$$

A simple inductive argument shows that

$$I(G)\,L^{(n)} \subseteq I^n(G)I(H) + I^{n-1}(G)I(G') + \ldots + I(G)I(G_n) + I(G_{n+1}) + L^{(n+1)},$$

and, therefore,

$$I^{n+1}(G) = I^n(G)I(H) + I^{n-1}(G)I(G') + \ldots + I(G)I(G_n) + I(G_{n+1}) + L^{(n+1)}.$$

Thus (4.3) holds for all $n \geqslant 2$. An application of the map $sa \to s$, $s \in S$, $a \in H$, when extended by linearity to ZG shows that

$$(I^{n-1}(G)I(H) + I^{n-2}(G)I(G') + \ldots + I(G)I(G_{n-1}) + I(G_n)) \cap L^{(n)} = 0.$$

Theorem 4.4 *Let G be a group with a normal subgroup H such that G/H is free abelian. Then*

$$G \cap (1 + I^n(G) + I(G)I(H)) = G_n H', \text{ for all } n \geqslant 1.$$

Proof. The result being trivially true for $n = 1, 2$, we may assume that $n \geqslant 3$. Then

$$G \cap (1 + I^n(G) + I(G)I(H)) \subseteq G \cap (1 + I^2(G)) = G' \subseteq H.$$

Let S be a left transversal of H in G as in 4.1 and θ: $ZG \to ZH$ be the map given on the elements of G by $\theta(sa) = a$, $s \in S$, $a \in H$ and linearity. An application of the map θ (with the notations of 4.1) then shows that

$$\begin{aligned} G \cap (1 + I^n(G) + I(G)I(H)) &= H \cap (1 + I(G_n) + I(G)I(H) + L^{(n)}) \\ &= H \cap (1 + I(G_n) + I^2(H)) \\ &= G_n H'. \end{aligned}$$

Sandling [10] proved that Abelian-by-cyclic groups have dimension property. It also follows from the work of Gupta (([1], Proposition IV.3.1 and Theorem IV.3.3) that free Abelian-by-free Abelian groups have dimension property. Recently Gupta [2] has proved that finitely generated metabelian p-groups, p-odd, have dimension property. As a consequence of Theorem 4.4, we have

COROLLARY 4.5

Abelian-by-free Abelian groups have dimension subgroup property.

COROLLARY 4.6

([1], Theorem IV.3.3). *Let F be a free or a free nilpotent or a free solvable group. Then*

$$F \cap (1 + I^n(F) + I(F)I(F')) = F'' F_n.$$

COROLLARY 4.7

Let G be a group and H, K be subgroups of G with H normal, $G = HK$, $H \cap K \subseteq [H, K]$ and $H/[H, K]$ free Abelian. Then

$$G \cap (1 + I^2(G)I(H)) = (H \cap G')'H_3.$$

Proof. We know ([5], Lemma 2.1(ii)) that

$$I^2(G)I(H) \subseteq I^3(H) + I(K)I(H) + I([H, K])I(H).$$

Therefore,

$$\begin{aligned} G \cap (1 + I^2(G)I(H)) &= H \cap (1 + I^2(G)I(H)) \\ &\subseteq H \cap (1 + I^3(H) + I(K)I(H) + I([H, K])I(H)) \\ &= H \cap (1 + I^3(H) + I([H, K])I(H)) \\ &= [H, K]'H_3 \text{ (by Theorem 4.4)} \end{aligned}$$

The reverse inclusion is clear. Also $[H, K]'H_3 = (H \cap G')'H_3$.

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On certain Ramanujan's mock theta functions

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Abstract. Ramanujan's last gift to the mathematicians was his ingeneous discovery of the mock theta functions of order three, five and seven. Recently, Andrews and Hickerson found a set of seven more functions in Ramanujan's Lost Note Book and formally labelled them as mock theta functions of order six. In this paper the complete forms of these functions have been studied and connected with the bilateral basic hypergeometric series ${}_2\Psi_2$. Several other interesting properties and transformations have also been studied.

Keywords. Mock theta functions; basic hypergeometric series; bilateral series.

1. Introduction

Ramanujan's last gift to the mathematicians was his ingeneous discovery of the mock theta functions of order three, five and seven. They were first studied by Watson [23]. Andrews [5, 6], in 1966, proved Ramanujan's identities for the third and fifth order mock theta functions by means of certain general theorems. Agarwal [1] in 1968, observed that the results of Andrews are not isolated ones but belong to a more general class of basic hypergeometric identities. Later, Andrews and his coworkers in a series of remarkable papers [5–10, 12, 13] derived Ramanujan's results and conjectures by the use of hypergeometric transformations and generalized Lambert's series. Agarwal [4], recently, suggested that it would be nice to study the mock theta functions in a unified manner through transformations of basic hypergeometric series and also try to prove the numerous identities of Ramanujan in the spirit "how Ramanujan might have proved them" instead of using techniques with which Ramanujan might not have been conversant. Andrews and Hickerson [13] in a recent paper in 1991 proved eleven other identities found in Ramanujan's 'Lost' Notebook and labelled the seven functions involved in these identities as the sixth order mock theta functions.

In § 3 of this paper, we study these seven sixth order mock theta functions further by defining their three 'complete' forms which throw more light on their structures and forms and subsequently derive interesting relations among these 'complete' sixth order mock theta functions and the sixth order functions.

We observe that all the mock theta functions of the third, fifth, seventh and the sixth order are simple limiting cases of $2\phi_1$'s, $3\phi_2$'s and $4\phi_3$'s basic hypergeometric series, on single bases (see for details Agarwal [4a]).

The definitions of sixth order functions as limiting cases of the ${}_3\phi_2$ functions (with q as one of its numerator parameters) suggest the study of this function on the pattern of the study made by Fine [15] for the Ramanujan's function $-\sum_{0}^{\infty}\frac{(a)_n t^n}{(b)_n}$. In §4 we derive three transformations connecting the sixth order mock theta functions, in a very straightforward manner, through the transformations of the special types of ${}_2\Psi_2$ and ${}_3\phi_2$ series studied earlier [18].

Finally, in §5 the complete form of the $\gamma(q)$ function, the last of the sixth order mock theta function has been studied. Andrews and Hickerson [13] have remarked that they could not find the pair for this function on the pattern that exists for the other six sixth order functions given by Ramanujan. We have suggested that the function $\chi(q)$ may be justifiably called the missing pair of the $\gamma(q)$-function. Some of the salient features of this paper were reported in the *Nat. Acad. Sci. Letters*, 15 (1992), 49–50.

2. Notations and results used

For $|q| < 1$, let

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}); \quad n \geqslant 1.$$

$$(a;q)_0 = 1, (a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j).$$

We, then define a generalized basic hypergeometric series, for $|q| < 1$, as:

$${}_A\Phi_{A-1}\left[\begin{matrix} a_1, a_2, \ldots, a_A \\ b_1, b_2, \ldots, b_{A-1}\end{matrix}; z\right] = \sum_{n=0}^{\infty}\frac{(a_1;q)_n\cdots(a_A;q)_n z^n}{(q;q)_n(b_1;q)_n\cdots(b_{A-1};q)_n}.$$

The series converges for $|z| < 1$. We denote $(a;q)_n$ simply by $(a)_n$ whenever there is no chance of any confusion with any other base. The symbol ${}_A\Phi_B^{(q^2)}$ denotes the generalized basic hypergeometric series on base q^2.

A generalized basic bilateral hypergeometric series is defined by

$${}_B\Psi_B\left[\begin{matrix} a_1, \ldots, a_B \\ b_1, \ldots, b_B\end{matrix}; z\right] = \sum_{n=\infty}^{\infty}\frac{(a_1;q)_n\cdots(a_B;q)_n z^n}{(b_1;q)_n\cdots(b_B;q)_n}$$

where $|b_1 b_2\cdots b_B/a_1 a_2\cdots a_B z| < 1$, $|z| < 1$, for convergence and

$$(a;q)_{-n} = \frac{(-)^n q^{n(n+1)/2}}{a^n(q/a;q)_n}.$$

Andrews and Hickerson [13] have defined the following seven sixth order mock theta functions as found in the 'Lost' Notebook of Ramanujan:

$$\phi(q) = \sum_{n=0}^{\infty}\frac{(-)^n q^{n^2}(q;q^2)_n}{(-q)_{2n}}, \tag{1}$$

$$\Psi(q) = \sum_{n=0}^{\infty}\frac{(-)^n q^{(n+1)^2}(q;q^2)_n}{(-q)_{2n+1}}, \tag{2}$$

$$\rho(q)=\sum_{n=0}^{\infty}\frac{q^{1/2n(n+1)}(-q)_n}{(q;q^2)_{n+1}}, \tag{3}$$

$$\sigma(q)=\sum_{n=0}^{\infty}\frac{q^{1/2(n+1)(n+2)}(-q)_n}{(q;q^2)_{n+1}}, \tag{4}$$

$$\lambda(q)=\sum_{n=0}^{\infty}\frac{(-)^n q^n(q;q^2)_n}{(-q)_n}, \tag{5}$$

$$\mu(q)=\sum_{n=0}^{\infty}\frac{(-)^n(q;q^2)_n}{(-q)_n}, \tag{6}$$

$$\gamma(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(q)_n}{(q^3;q^3)_n}. \tag{7}$$

It may be noted that $\mu(q)$ in (6) represents a divergent series. As remarked by Andrews [13] one gives it a meaning as "the sequence of even partial sums converge, as does the sequences of odd partial sums; we define $\mu(q)$ as the average of these two values."

Indeed one could see this fact more clearly if one uses the transformation

$$\sum_{n=0}^{\infty}\frac{(\alpha;q^2)_n(\beta)_n\tau^n}{(q^2;q^2)_n(\gamma)_n}=\frac{(\beta)_\infty(\alpha\tau;q^2)_\infty}{(\gamma)_\infty(\tau;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(\gamma/\beta)_{2m}(\tau;q^2)_m\beta^{2m}}{(q)_{2m}(\alpha\tau;q^2)_m}+$$
$$+\frac{(\beta)_\infty(\alpha\tau q;q^2)_\infty}{(\gamma)_\infty(\tau q;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(\gamma/\beta)_{2m+1}(\tau q;q^2)_m\beta^{2m+1}}{(q)_{2m+1}(\alpha\tau q;q^2)_m},$$

obtained from Theorem A_1 of Andrews [5; p. 67].

Taking $\alpha=q$, $\beta=q$ and $\gamma=0$, $\tau=-1$, the LHS of the above eqn becomes $\mu(q)$ and the RHS becomes a sum of two convergent series with even and odd powers of β.

3. The 'complete' sixth order mock theta functions

(i) Let us consider the bilateral series

$$\begin{aligned}F_c(\alpha)&=\sum_{-\infty}^{\infty}\frac{(-)^n q^{n^2}\alpha^{2n}(q;q^2)_n}{(-\alpha q)_{2n}}\\&=\sum_{0}^{\infty}\frac{(-)^n q^{n^2}\alpha^{2n}(q;q^2)_n}{(-\alpha q)_{2n}}+\sum_{1}^{\infty}\frac{(-)^n q^{n^2}\alpha^{-2n}(q;q^2)_{-n}}{(-\alpha q)_{-2n}}\\&=\sum_{0}^{\infty}\frac{(-)^n q^{n^2}\alpha^{2n}(q;q^2)_n}{(-\alpha q)_{2n}}+\frac{q(1+1/\alpha)(1+q/\alpha)}{1-q}\sum_{0}^{\infty}\frac{q^n(-q^2/\alpha;q)_{2n}}{(q^3;q^2)_n}.\end{aligned} \tag{8}$$

In (8) taking $\alpha=1$ and q, respectively, we get

$$F_c(1)\equiv\phi_c(q)=\phi(q)+\frac{2q(1+q)}{1-q}\,{}_3\phi_1^{(q^2)}\left[\begin{matrix}-q^2,-q^3,q^2\\q^3\end{matrix};q\right], \tag{9}$$

where $\phi_c(q)$ denotes the 'complete' or 'bilateral' $\phi(q)$ function (1) and

$$\frac{q}{1+q}F_c(q) \equiv \Psi_c(q) = \Psi(q) + \frac{2q}{1-q}\,{}_3\phi_1^{(q^2)}\left[\begin{matrix}-q,-q^2,q^2\\ q^3\end{matrix};q\right] \tag{10}$$

where $\Psi_c(q)$ denotes the 'complete' or 'bilateral' $\Psi(q)$ function (2).
(ii) Next, consider the function $R_c(\alpha) =$

$$\sum_{-\infty}^{\infty}\frac{q^{n(n+1)/2}(-q)_n\alpha^n}{(q;q^2)_{n+1}} =$$

$$= \sum_{0}^{\infty}\frac{q^{n(n+1)/2}(-q)_n\alpha^n}{(q;q^2)_{n+1}} + \frac{1}{2\alpha}\sum_{0}^{\infty}\frac{(-)^n q^n\alpha^{-n}(q;q^2)_n}{(-q;q)_n}. \tag{11}$$

In (11), taking $\alpha = 1$, we get from (3) and (5),

$$R_c(1) = \rho_c(q) = \rho(q) + \tfrac{1}{2}\lambda(q). \tag{12}$$

In (11), taking $\alpha = q$, we get from (4) and (6)

$$qR_c(q) = \sigma_c(q) = \sigma(q) + \tfrac{1}{2}\mu(q); \tag{13}$$

where $\rho_c(q)$ and $\sigma_c(q)$ are, respectively, 'complete' forms for $\rho(q)$ and $\sigma(q)$.
(iii) Again, consider the function $M_c(\alpha) =$

$$\sum_{-\infty}^{\infty}\frac{(-)^n\alpha^n(q;q^2)_n}{(-q)_n} =$$

$$= \sum_{0}^{\infty}\frac{(-)^n\alpha^n(q;q^2)_n}{(-q)_n} + \frac{2}{\alpha}\sum_{0}^{\infty}\frac{\alpha^{-n}(-q;q)_n q^{(n+1)(n+2)/2}}{(q;q^2)_{n+1}}. \tag{14}$$

For $\alpha = 1, q$, respectively in (14), we get from (4) and (6)

$$M_c(1) \equiv \mu_c(q) = \mu(q) + 2\sigma(q); \tag{15}$$

$$M_c(q) \equiv \lambda_c(q) = \lambda(q) + 2\rho(q); \tag{16}$$

where $\mu_c(q)$ and $\lambda_c(q)$, respectively, are the 'complete' $\mu(q)$ and $\lambda(q)$ functions.
From (12) and (16), we get, the surprisingly curious equivalence

$$\lambda_c(q) = 2\rho_c(q). \tag{17}$$

Also, from (13) and (15), we get

$$\mu_c(q) = 2\sigma_c(q). \tag{18}$$

(17) and (18) can also be obtained from the following transformation of Bailey [14];

$${}_2\Psi_2\left[\begin{matrix}a,b\\ c,d\end{matrix};z\right] = \frac{(az,bz,cq/abz,dq/abz;q)_\infty}{(q/a,q/b,c,d;q)_\infty}\,{}_2\Psi_2\left[\begin{matrix}abz/c, abz/d\\ az, \quad bz\end{matrix};\frac{cd}{abz}\right]. \tag{19}$$

In (19) taking $a = \alpha$, $b = -q$, $c = q^{3/2}$, $d = -q^{3/2}$, $z = -q/\alpha$ and then letting $\alpha \to \infty$, we again obtain (17).

Next, in (19) taking $a=\alpha$, $b=-q$, $c=q^{3/2}$, $d=-q^{3/2}$, $z=-q^2/\alpha$ and then letting $\alpha\to\infty$, we again obtain (18).

Again in (19) taking $q\equiv q^2$, $a=\alpha$, $b=q$, $c=-q^2$, $d=-q$, $z=q/\alpha$ and then letting $\alpha\to\infty$, we obtain (9).

Finally, in (19) taking $q\equiv q^2$, $a=\alpha$, $b=q$, $c=q^2$, $d=-q^3$, $z=q^3/\alpha$ and then letting $\alpha\to\infty$, we obtain (10).

The above analysis also shows that the sixth order mock theta functions can be conveniently studied through the known transformation theory of ${}_2\Psi_2$ functions.

If one uses the known identities $(0{\cdot}18)_R-(0{\cdot}21)_R$ of Ramanujan, proved by Andrews and Hickerson [13], it can be easily seen that the various 'complete' mock theta functions of order six can be reexpressed as

$$\phi_c(q)+2\sigma(q^{1/2})=\frac{2q(1+q)}{1-q}\,{}_3\phi_1^{(q^2)}\left[\begin{matrix}-q^2,-q^3,q^2\\ q^3\end{matrix};q\right]+ \\ +(-q^{1/2};q)_\infty^2\,j(-q^{3/2};q^3). \tag{20}$$

$$\phi_c(q)-\mu(q^{1/2})=\frac{2q(1+q)}{1-q}\,{}_3\phi_1(q^2)\left[\begin{matrix}-q^2,-q^3,q^2\\ q^3\end{matrix};q\right]+ \\ +\tfrac{1}{2}(q^{1/2};q)_\infty^2\,j(q^{3/2};q^3). \tag{21}$$

$$\Psi_c(q)+q^{1/2}\rho(q^{1/2})=\frac{2q}{1-q}\,{}_3\phi_1^{(q^2)}\left[\begin{matrix}-q,-q^2,q^2\\ q^3\end{matrix};q\right]+ \\ +q^{1/2}(-q^{1/2};q)_\infty^2\,j(-q^{1/2};q^3). \tag{22}$$

$$\Psi_c(q)+\frac{q^{1/2}}{2}\lambda(-q^{1/2})=\frac{2q}{1-q}\,{}_3\phi_1^{(q^2)}\left[\begin{matrix}-q,-q^2,q^2\\ q^3\end{matrix};q\right]+ \\ +\frac{q^{1/2}}{2}(-q^{1/2};q)_\infty^2\,j(-q^{1/2};q^3). \tag{23}$$

$$\rho_c(q)=\tfrac{1}{2}\lambda_c(q)\quad=\tfrac{1}{2}(q;q^2)_\infty^2\,j(q;q^6)+ \\ +(-q;q^2)_\infty^2\,j(-q;q^6). \tag{24}$$

$$\rho_c(q)=\tfrac{1}{2}\lambda(q)+\lambda(-q)-q^{-1}\Psi(q^2). \tag{25}$$

$$\sigma_c(q)=\tfrac{1}{2}\mu_c(q)=\tfrac{1}{2}(-q;q^2)_\infty^2\,j(-q^3;q^6)-\frac{1}{4}(q;q^2)_\infty^2\,j(q^3;q^6). \tag{26}$$

$$\sigma_c(q)=\tfrac{1}{2}\mu(q)-\mu(-q)+\tfrac{1}{2}\phi(q^2), \tag{27}$$

where $j(\alpha;q^k)=(\alpha,q^k/\alpha,q^k;q^k)_\infty$.

Next let us consider another bilateral transformation of Bailey [14], namely

$${}_2\Psi_2\left[\begin{matrix}a,b\\ c,d\end{matrix};z\right]=\frac{(az,d/a,c/b,dq/abz;q)_\infty}{(z,d,q/b,cd/abz;q)_\infty}\,{}_2\Psi_2\left[\begin{matrix}a,abz/d;\\ az,c\end{matrix};d/a\right]. \tag{28}$$

In (28), changing the base from q to q^2 and putting $a=t$, $b=q$, $c=-q$, $d=-q^2$,

$z = q/t$ and then letting $t \to \infty$, we get

$$\frac{(q;q^2)_\infty}{(-1;q^2)_\infty}\phi_c(q) = \sum_{-\infty}^{\infty} \frac{q^{n(n+1)}(-1;q^2)_n}{(q;q^2)_n(-q;q^2)_n}, \tag{29}$$

which shows that $\dfrac{(q;q^2)_\infty \phi_c(q)}{(-1;q^2)_\infty}$ is an even function of q.

In (28), also changing the base from q to q^2 and putting $a = \alpha$, $b = q$, $c = -q^2$, $d = -q^3$, $z = q^3/\alpha$ and then letting $\alpha \to \infty$, we get

$$\Psi_c(-q) = -\frac{(q)_\infty}{(-q)_\infty}\Psi(q) + \frac{2q}{1-q}\,{}_3\phi_1^{(q^2)}\left[\begin{matrix}-q,-q^2,q^2\\ q^3\end{matrix};q\right]. \tag{30}$$

Comparing (30) with (10) we obtain that $(q)_\infty \psi_c(q)$ is an odd function of q.

It may be interesting to compare this with a similar phenomenon in the case of the 'complete' series for the $f_0(q)$, $\Psi_0(q)$, $f_1(q)$, and $\psi_1(q)$, the four fifth order mock theta functions, as remarked by Watson [23; section VII, p. 296].

4. The sixth order mock theta functions and three families of generalized functions associated with them

It may be observed that the third, fifth and seventh order mock theta functions are expressible in terms of a ${}_2\phi_1$, ${}_3\phi_2$ and a ${}_4\phi_3$ series, respectively. This seemingly simple observation has far reaching ramifications and has in it the germs of the basis of classification of the mock theta functions of Ramanujan, as of order three, five and seven, a fact which neither Ramanujan nor anybody else during the past seven decades has, thus far, implicity or explicitly defined. (For details see [4a] and [18a]).

In particular, the seven "sixth" order mock theta functions are easily seen to be expressible as follows;

$$\phi(q) = \mathop{Lt}_{\alpha\to 0}\,{}_3\phi_2^{(q^2)}\left[\begin{matrix}1/\alpha, q, q^2\\ -q^2, -q\end{matrix};q\alpha\right]; \tag{31}$$

$$\Psi(q) = \frac{q}{(1+q)}\mathop{Lt}_{\alpha\to 0}\,{}_3\phi_2^{(q^2)}\left[\begin{matrix}1/\alpha, q, q^2\\ -q^2, -q^3\end{matrix};q^3\alpha\right]; \tag{32}$$

$$\rho(q) = \frac{1}{1-q}\mathop{Lt}_{\alpha\to 0}\,{}_3\phi_2\left[\begin{matrix}1/\alpha, q, -q\\ q^{3/2}, -q^{3/2}\end{matrix};-q\alpha\right]; \tag{33}$$

$$\sigma(q) = \frac{q}{1-q}\mathop{Lt}_{\alpha\to 0}\,{}_3\phi_2\left[\begin{matrix}1/\alpha, -q, q\\ q^{3/2}, -q^{3/2}\end{matrix};-q^2\alpha\right], \tag{34}$$

$$\lambda(q) = {}_3\phi_2\left[\begin{matrix}q^{1/2}, -q^{1/2}, q\\ -q, 0\end{matrix};-q\right] \tag{35}$$

$$\mu(q) = {}_3\phi_2\left[\begin{matrix}q^{1/2}, -q^{1/2}, q;\\ -q, 0\end{matrix}-1\right]; \tag{36}$$

$$\gamma(q) = \underset{\alpha,\beta\to 0}{Lt}\ {}_3\phi_2\left[\begin{matrix}1/\alpha, 1/\beta, q \\ q\omega, q\omega^2\end{matrix}; q\alpha\beta\right]. \tag{37}$$
$$= (1-\omega^2)\,{}_2\phi_1[0, q; \omega q; \omega^2],$$

on using a well-known transformation due to Sears [20], and where ω is a primitive cube root of unity.

The above forms of definitions of the 'sixth' order functions suggest the study of transformations of the function ${}_3\phi_2(q, a, b; c, d; t)$ and its limiting cases.

We now proceed to prove that for $\alpha = q^r$, $r = 0, 1, 2, \ldots$ the following three functions $A(\alpha, q)$, $B(\alpha, q)$ and $C(\alpha, q)$ form a family of functions closely associated with mock theta functions of order six. The three functions are defined as

$$A(\alpha; q) = \sum_{n=0}^{\infty} \frac{(-)^n q^{n^2} \alpha^{2n} (q; q^2)_n}{(-\alpha q; q)_{2n}}; \tag{38}$$

$$B(\alpha; q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} \alpha^n (-q)_n}{(q; q^2)_{n+1}}; \tag{39}$$

$$C(\alpha; q) = \sum_{n=0}^{\infty} \frac{(-)^n \alpha^n (q; q^2)_n}{(-q)_n}. \tag{40}$$

To prove that (38) gives a family of functions, associated with the mock theta functions $\phi(q)$ and $\Psi(q)$, take $q \equiv q^2$, $a = \alpha$, $b = q$, $c = -q^2$, $d = -q$, $z = q/\alpha$ and let $\alpha \to 0$, in (19), to obtain, after some simplification,

$$\phi(q) + \Psi(q) - (1+q)q^2 \sum_{n=0}^{\infty} \frac{(-)^n q^{n^2+4n} (q; q^2)_n}{(-q; q)_{2n+2}} - 1 = 0.$$

In terms of $A(\alpha, q)$, we have

$$A(1; q) + \frac{q}{1+q} A(q; q) - \frac{q^2}{1+q^2} A(q^2; q) - 1 = 0. \tag{41}$$

Since $A(1; q)$ and $A(q; q)$ are known to be mock theta functions it follows on repeating the argument, that $A(q^r; q)$, $r = 0, 1, 2, \ldots$ is an infinite family of functions connected with $\phi(q)$ and $\Psi(q)$.

Next, consider a transformation between three ${}_3\Phi_2$'s derived by me in an earlier paper [18], namely

$$q^2(1-t)\,{}_3\Phi_2[a, b, q; c, d; t] + [(a+b)tq^2 - (c+d)q]\,{}_3\Phi_2[a, b, q; c, d; tq] +$$
$$+ [cd - abtq^2]\,{}_3\Phi_2[a, b, q; c, d; tq^2] + (c+d-q)q - cd = 0. \tag{42}$$

In (42), taking $a = \alpha$, $b = -q$, $c = q^{3/2}$, $d = -q^{3/2}$, $t = -q/\alpha$, letting $\alpha \to \infty$ and using (33) and (34), we get

$$\rho(q) - \sigma(q) - q(1+q) \sum_{0}^{\infty} \frac{q^{n^2/2 + 5n/2} (-q)_n}{(q; q^2)_{n+1}} - 1 = 0.$$

In terms of the $B(\alpha; q)$ functions, we have

$$B(1;q) - qB(q;q) - q(1+q)B(q^2;q) - 1 = 0. \tag{43}$$

As in (41) it follows that $B(q^r;q)$, $r = 0, 1, 2, \ldots$ defines an infinite family of functions associated with $\rho(q)$ and $\sigma(q)$. Lastly, in (42) taking $a = q^{1/2}$, $b = -q^{1/2}$, $c = -q$, $d = 0$, $t = -1$ and using (35) and (36), we get

$$2\mu(q) + \lambda(q) - q\sum_{0}^{\infty} \frac{(-)^n q^{2n}(q;q^2)_n}{(-q)_n} - 2 = 0.$$

In terms of $C(\alpha, q)$ this can be rewritten as

$$2C(1;q) + C(q;q) - qC(q^2;q) - 2 = 0. \tag{44}$$

Therefore it follows that $C(q^r;q)$ defines an infinite family of functions associated with $\mu(q)$ and $\lambda(q)$.

The functions $A(\alpha;q)$, $B(\alpha;q)$ and $C(\alpha;q)$ are not without interest. As for instance, for $\alpha = -1$, we have

$$A(-1;q) = \frac{(q^2, q^4, q^6; q^6)_\infty (q;q^2)_\infty}{(q^2;q^2)_\infty}, \tag{45}$$

on using an identity due to Slater [21;(23)].

An identity of Ramanujan found in the 'Lost' notebook [11;(3.17)$_R$], namely,

$$\begin{aligned} q + \sum_{1}^{\infty} \frac{q^{n+1}(-aq)_n(-bq)_n}{(-cq)_n} &= c/ab \sum_{1}^{\infty} \frac{q^{n(n+1)/2} a^n b^n c^{-n}(-1/c)_n}{(aq/c)_n(bq/c)_n} \\ &\quad - c/ab \frac{(-aq)_\infty(-bq)_\infty}{(-cq)_\infty} \sum_{1}^{\infty} \frac{q^{n^2} a^n b^n c^{-2n}}{(aq/c)_n(bq/c)_n}, \end{aligned} \tag{46}$$

for $a = -q^{-1/2}$, $b = q^{-1/2}$, $c = 1$ connects $C(-q;q)$ with $B(-1;q)$ in the form

$$qC(-q;q) = 2qB(-1;q) + \frac{q(q;q^2)_\infty}{(-q)_\infty} \sum_{1}^{\infty} \frac{(-)^n q^{n(n-1)}}{(q;q^2)_n}. \tag{47}$$

5. The 'complete' mock theta function $\gamma_c(q)$, $\gamma(q)$ and its missing pair function

While searching for the pair function corresponding to the last of the 'sixth' order mock theta function $\gamma(q)$, we first derive some simple alternative forms for the $\gamma(q)$ function. We have

$$\begin{aligned} \gamma(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}(q)_n}{(q^3;q^3)_n} \\ &= \underset{\alpha,\beta\to 0}{Lt}\ {}_3\Phi_2\left[\begin{matrix} 1/\alpha, 1/\beta, q \\ q\omega, q\omega^2 \end{matrix}; q\alpha\beta\right], \\ &= (1-\omega^2)\,{}_2\Phi_1[0, q; \omega q; \omega^2], \end{aligned} \tag{48}$$

where ω is a primitive cube root of unity.

Using the following ${}_3\Phi_2$ transformation of Sears [20], namely,

$$\sum_{n=0}^{\infty}\frac{(e)_n(f)_n(qbc/ef)^n}{(qb)_n(qc)_n}=\frac{(qbc/f)_\infty(qc/e)_\infty}{(qc)_\infty(qbc/ef)_\infty}\times$$
$$\times\sum_{n=0}^{\infty}\frac{(b)_n(e)_n(qb/f)_n(qc/e)^n}{(q)_n(qbc/f)_n(qb)_n}. \tag{49}$$

In (49) taking $e=1/\alpha$, $f=1/\beta$, $b=\omega$, $c=\omega^2$ and letting α and $\beta\to 0$, we get

$$\gamma(q)=\frac{1}{(q\omega^2)_\infty}\sum_{n=0}^{\infty}\frac{(-)^n(\omega)_n\omega^{2n}q^{n(n+1)/2}}{(q)_n(q\omega)_n},$$

where ω is a primitive cube root of unity.

Again, use another transformation of Sears [20], namely

$$\sum_{n=0}^{\infty}\frac{(qb/e)_n(qb/f)_n(q/a)_nc^n}{(q)_n(qb/a)_n(q^2bc/ef)_n}$$
$$=\frac{(qc/a)_\infty(qabc/ef)_\infty}{(q^2bc/ef)_\infty(c)_\infty}\sum_{n=0}^{\infty}\frac{(e/a)_n(f/a)_n(q/a)_n(qabc/ef)^n}{(q)_n(qb/a)_n(qc/a)_n}. \tag{50}$$

In (50) first putting $a=1$, then taking $e=1/\alpha$, $f=1/\beta$, $b=\omega$, $c=\omega^2$ and letting α and $\beta\to 0$, we get

$$\gamma(q)=(1-\omega^2)\sum_{n=0}^{\infty}\frac{\omega^{2n}}{(\omega q)_n}, \tag{51}$$

where ω is a primitive cube root of unity.

Also, one can easily obtain from the result (16.4) of Fine [15] another representation for $\gamma(q)$ given by Andrews and Hickerson [13, (2.25)], without going through the various steps adopted by them. In (16.4) of Fine [15], taking $b=\omega$ and $t=\omega^2$, we get

$$\gamma(q)=(1-\omega^2)\,{}_2\Phi_1(0,q;\omega q;\omega^2)$$
$$=\frac{1}{(q)_\infty}\left\{1+3\sum_{n=1}^{\infty}\frac{(-)^n(1+q^n)q^{(3n^2+n)/2}}{1+q^n+q^{2n}}\right\}$$

or

$$(q)_\infty\gamma(q)=\sum_{-\infty}^{\infty}\frac{3(-1)^nq^{n(3n+1)/2}}{1+q^n+q^{2n}}, \tag{52}$$

which is the result of Andrews and Hickerson [13, (2.25)].

From (48), we observe that the definition of $\gamma(q)$ suggests a similar form for $\gamma_1(q)$, the pair function of $\gamma(q)$, i.e.,

$$\gamma_1(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q)_n}{(-q^3;q^3)_n}$$
$$=(1+\omega^2)\,{}_2\phi_1(0,q;-\omega q;-\omega^2). \tag{53}$$

One easily sees that $\gamma_1(q)$ is the third order mock theta function $\chi(q)$.

We, now, define a 'complete' mock theta function $G_c(\alpha)$ by the bilateral sereis

$$\sum_{-\infty}^{\infty} \frac{q^{n^2}(\alpha q)_n}{(\alpha^3 q^3; q^3)_n}.$$

We have

$$\sum_{-\infty}^{\infty} \frac{q^{n^2}(\alpha q)_n}{(\alpha^3 q^3; q^3)_n}$$

$$= \sum_{0}^{\infty} \frac{q^{n^2}(\alpha q)_n}{(\alpha^3 q^3; q^3)_n} + \frac{\alpha^3 q(1 - 1/\alpha^3)}{(1 - 1/\alpha)} \sum_{0}^{\infty} \frac{(\alpha^2 q)^n (q^3/\alpha^3; q^3)_n}{(q/\alpha; q)_n}. \tag{54}$$

In (54), taking $\alpha = 1$, we get

$$G_c(1) \equiv \gamma_c(q) = \gamma(q) + 3q \sum_{0}^{\infty} \frac{q^n (q^3; q^3)_n}{(q; q)_n}, \tag{55}$$

where $\gamma_c(q)$ denotes the 'complete' $\gamma(q)$ function.
Again, in (54) taking $\alpha = -1$, we get

$$G_c(-1) \equiv \chi_c(q) = \chi(q) + q \sum_{0}^{\infty} \frac{q^n (-q^3; q^3)_n}{(-q; q)_n}, \tag{56}$$

where $\chi_c(q)$ denotes the 'complete' $\chi(q)$ function.
In the result of Fine [15, (16.4)] taking $t = -\omega^2$, we get

$$\chi(q) = (1 + \omega^2)\, {}_2\phi_1(0, q; -\omega q; -\omega^2)$$

$$= \frac{1}{(q)_\infty} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-)^n (1 + q^n) q^{(3n^2 + n)/2}}{1 - q^n + q^{2n}} \right\}$$

or

$$(q)_\infty \chi(q) = \sum_{-\infty}^{\infty} \frac{(-)^n q^{n(3n+1)/2}}{1 - q^n + q^{2n}}. \tag{57}$$

In the 'Lost' Note book on p. 17, the last but one formula mentions a relation between $\gamma(-q)$, $\chi(-q)$, the third order mock theta function. This indicates that probably $\gamma(q)$ belongs to the class of the third order mock theta functions.

6. Conclusion

The above analysis in §§3, 4 and 5 and the works of earlier authors makes it very clear that for a better understanding of the structure and transformation theory of the various mock theta functions, perhaps, the best tool is the corresponding basic bilateral hypergeometric series or their 'complete' forms (See Agarwal [4a] and Gupta [18a]).

I think it may also be possible to deduce some of the relations derived in this paper by using series manipulation of the Lambert series representations of the functions of Andrews and Hickerson [13].

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Lambert series and Ramanujan

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Abstract. Lambert series are of frequent occurrence in Ramanujan's work on elliptic functions, theta functions and mock theta functions. In the present article an attempt has been made to give a critical and up-to-date account of the significant role played by Lambert series and its generalizations in further development and a better understanding of the works of Ramanujan in the above and allied areas.

Keywords. Basic hypergeometric series; Lambert series; elliptic functions; mock theta functions.

1. Introduction

The topic of this article called the *Lambert series* is a very well-known class of series, both as analytic function and number theories. Our present interest in these series emerges out of the works of Ramanujan who has profusely and elegantly used these series in a variety of contexts in his mystic research work. A number of mathematicians, later, in an attempt to unravel the mysticism underlying some of the unproved identities of Ramanujan, have also used these series with advantage.

To appreciate the effectiveness of choice of the use of these series by Ramanujan and subsequently, by other mathematicians in proving a variety of identities, one must examine how beautifully they enter into the theory of numbers, theory of Weierstrass's elliptic functions and the basic hypergeometric theory.

We, now present our work starting with the definition and then the impact of Lambert series on the development of the above areas.

2. Lambert series

The series

$$\sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}, \tag{2.1}$$

was considered by Lambert [19] in connection with the convergence of power series. If the series $\sum_{1}^{\infty} a_n$ converges, then the Lambert series (2.1) converges for all values of

x except for $x = \pm 1$; otherwise it converges for those values of x for which the series $\sum_{1}^{\infty} a_n x^n$ converges. The Lambert series is used in certain problems of number theory. Thus, for $|x| < 1$ the sum $\phi(x)$ of the series (2.1) can be represented as a power series

$$\phi(x) = \sum_{1}^{\infty} \alpha_n x^n \tag{2.2}$$

where

$$\alpha_n = \sum_{k|n} a_k,$$

and the summation is over all divisors k of n. In particular, if $a_n = 1$, then $\alpha_n = \tau(n)$, the number of divisors of n. The behaviour of $\phi(x)$ (with suitable a_n) as $x \uparrow 1$ is used (see for example [20]) in the problem of Hardy and Ramanujan on obtaining an asymptotic formula for the number of unbounded partitions of a natural number.

Lambert series also occur in the expansion of Eisenstein series, a particular kind of modular form.

We list below a variety of familiar functions which can be expressed in terms of a Lambert series [18, pp. 448–451]:

(i) If $a_n = \mu(n)$, where $\mu(n)$ is the Möbius function, then

$$z = \sum_{1}^{\infty} \mu(n) \frac{z^n}{1 - z^n}.$$

(ii) For $|z| < 1$, $\dfrac{z}{(1-z)^2} = \sum_{1}^{\infty} \phi(n) \dfrac{z^n}{1 - z^n}$,

where $\phi(n)$ is the Euler function defined as the number of positive integers not exceeding n that are relatively prime to n.

(iii) If $\sum_{n=1}^{\infty} a_n z^n/1 - z^n = f(z)$ and $\sum_{n=1}^{\infty} a_n z^n = g(z)$, then one can easily see that

$$f(z) = \sum_{m=1}^{\infty} g(z^m).$$

(iv) For $a_n = (-1)^{n-1}$; $a_n = n$; $a_n = (-1)^{n-1} n$; $a_n = 1/n$; $a_n = (-1)^{n-1}/n$; $a_n = \alpha^n$; we have for $|z| < 1$, the remarkable identities, (each summation is for $n = 1$ to ∞),

a) $$\sum (-)^{n-1} \frac{z^n}{1 - z^n} = \sum \frac{z^n}{1 + z^n}$$

b) $$\sum n \frac{z^n}{1 - z^n} = \sum \frac{z^n}{(1 - z^n)^2}$$

c) $$\sum (-)^{n-1} \frac{n z^n}{1 - z^n} = \sum \frac{z^n}{(1 + z^n)^2}$$

d) $$\sum \frac{1}{n} \frac{z^n}{1 - z^n} = \sum \log\left(\frac{1}{1 - z^n}\right)$$

e) $\sum \frac{(-)^{n-1}}{n}\frac{z^n}{1-z^n} = \sum \log(1+z^n)$

f) $\sum \alpha^n \frac{z^n}{1-z^n} = \sum \frac{\alpha z^n}{1-\alpha z^n}$,

respectively.

(v) In the identities (iv) (d) and (e) above, with the series of Logarithms (for which we take the principal values) one easily sees that they are equivalent to

$$\prod_1^\infty (1-z^n) = e^{-\omega} \text{ with } \omega = \sum \frac{1}{n}\frac{z^n}{1-z^n},$$

$$\prod_1^\infty (1+z^n) = e^{\omega} \text{ with } \omega = \sum \frac{(-)^{n-1}}{n}\frac{z^n}{1-z^n}.$$

(vi) An interesting numerical example is furnished by the Fibonacci sequence of numbers defined by $u_0 = 0$, $u_i = 1$ and for every $n > 1$, $u_n = u_{n-1} + u_{n-2}$, namely

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

We then have

$$\sum_{k=1}^{\infty} \frac{1}{u_{2k}} = 1 + \frac{1}{3} + \frac{1}{8} + \frac{1}{21} + \frac{1}{55} + \cdots$$
$$= \sqrt{5}\left[L\left(\frac{3-\sqrt{5}}{2}\right) - L\left(\frac{7-3\sqrt{5}}{2}\right)\right],$$

where $L(x)$ is the sum of Lambert series $\Sigma x^n/1 - x^n$. The proof is based on the fact that

$$u_\nu = \frac{\alpha^\nu - \beta^\nu}{\alpha - \beta}, \quad (\nu = 0, 1, 2, \ldots)$$

where α and β are the roots of the quadratic equation

$$x^2 - x - 1 = 0.$$

(vii) Lastly, an important illustration is provided by the identity

$$\log\Gamma_q(x) = -\log(1-q) + \log q \sum_{n=0}^{\infty} \frac{q^{n+x}}{1-q^{n+x}},$$

where $0 < q < 1$, $x > 0$ and $\Gamma_q(x)$ is Jackson's q-analogue of gamma function defined as $(q)_\infty(1-q)^{1-x}/(q^x)_\infty$.

The integral transform

$$F(x) = \int_0^\infty \frac{ta(t)}{e^{xt}-1}\,dt,$$

known as the Lambert transform is the continuous analogue of the Lambert series (under the correspondence $ta(t)\leftrightarrow a_n$, $e^{-x}\leftrightarrow x$). The following inversion formula holds: Suppose that

$$a(t)\in L(0,\infty)$$

and that

$$\lim_{t\to+0} a(t)t^{1-\delta}=0,\quad \delta>0.$$

If also $\tau>0$ and if the function $a(t)$ is continuous at $t=\tau$, then one has

$$\tau a(\tau)=\lim_{k\to\infty}\frac{(-1)^k}{k!}\left(\frac{k}{\tau}\right)^{k+1}\sum_{n=1}^{\infty}\mu(n)n^k F^{(k)}\left(\frac{nk}{\tau}\right),$$

where $\mu(n)$ is the Möbius function (Widder [30]).

Still another case of particular interest is the Lambert series obtained by taking

$$\begin{aligned}F(x)&=\sum_{n=1}^{\infty}a_n\frac{x^n}{1-x^n}\\&=\sum_{n=1}^{\infty}a_n\sum_{m=1}^{\infty}x^{mn}\\&=\sum_{N=1}^{\infty}b_Nx^N,\end{aligned}$$

where $b_N=\sum_{n|N}a_n$.

The relation between $\{a_n\}$ and $\{b_n\}$ is equivalent to

$$\xi(s)f(s)=g(s),$$

where $f(s)$ and $g(s)$ are respectively the Dirichlet series associated with $\{a_n\}$ and $\{b_n\}$. Hence, if $f(s)=\sum_1^{\infty}a_nn^{-s}$ and $g(s)=\sum_1^{\infty}b_nn^{-s}$, then

$$F(x)=\sum_{n=1}^{\infty}a_n\frac{x^n}{1-x^n}=\sum_{n=1}^{\infty}b_nx^n,$$

if and only if

$$\xi(s)f(s)=g(s).$$

3. Lambert series and certain arithmetical functions

We begin with an identity founded by E.T. Bell, namely

$$\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}\left[\frac{1}{1-q}+\frac{1}{1-q^2}+\cdots+\frac{1}{1-q^n}\right]=\sum_{n=1}^{\infty}\frac{n^2q^n}{1-q^n}. \tag{3.1}$$

Bell obtained the formula (or its arithmetical equivalent) while looking for quadratic forms which represent all integers with at most a finite number of exceptions. An immediate equivalent of (3.1) is

$$\sum_{s=1}^{n-1}\left[\sum_{d\mid s} N_d(n-s)\right] = \zeta_2(n) - \zeta_0(n), \tag{3.2}$$

where $N_a(m)$ is the sum of the divisors of m not greater than m/a, and $\zeta_r(n)$ denotes the sum of rth powers of the divisors of n. Another consequence of (3.1) mentioned by Bell is that the number of representations of any integer $n>0$ in the form $wx+xy+yz+zu$, where $w,x,z,u>0, y\geqslant 0$, is

$$\zeta_2(n) - n\zeta_0(n),$$

a classical result stated without proof by Liouville, in 1867.

Bailey [12] gave three proofs for (3.1). Two of the proofs are probably as elementary as one could expect.

In the first proof, he began by proving the formula

$$\begin{aligned}\frac{z}{1-q}+\frac{2z^2}{1-q^2}+\frac{3z^3}{1-q^3}+\cdots &= \frac{z}{(1-q)(1-z)}+\frac{(1-q)z^2}{(1-q^2)(1-z)(1-qz)}+\\ &\quad+\frac{(1-q)(1-q^2)z^3}{(1-q^3)(1-z)(1-qz)(1-q^2z)}+\cdots\end{aligned} \tag{3.3}$$

where $|z|<1$, $|q|<1$.

Denoting the right hand side by $F(z)$, and writing

$$(a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad (a)_0 = 1,$$

we have, on simplification

$$F(z) - F(qz) = \frac{z}{(1-z)^2}.$$

It follows that

$$\begin{aligned}F(z) &= \frac{z}{(1-z)^2}+\frac{qz}{(1-qz)^2}+\frac{q^2z}{(1-q^2z)^2}+\cdots\\ &= \sum_{r=0}^{\infty}\frac{q^rz}{(1-q^rz)^2} = \sum_{r=0}^{\infty}\sum_{n=1}^{\infty} n(q^rz)^n = \sum_{n=1}^{\infty}\frac{nz^n}{1-q^n},\end{aligned}$$

and this proves (3.3).

Now differentiating (3.3) with respect to z, we obtain the formula

$$\sum_{n=1}^{\infty}\frac{(q)_{n-1}z^n}{(1-q^n)(z)_n}\left[\frac{1}{1-z}+\frac{1}{1-qz}+\cdots+\frac{1}{1-q^{n-1}z}\right] = \sum_{n=1}^{\infty}\frac{n^2z^n}{1-q^n}, \tag{3.4}$$

which reduces to (3.1) when $z=q$.

Another result given by Bailey, in this context, was the identity

$$(1-a)\sum_{n=1}^{\infty}\frac{(1-aq^{2n})q^n}{(1-q^n)^2(1-aq^n)^2}\left[\frac{1}{1-q}+\frac{1}{1-q^2}+\cdots+\frac{1}{1-q^n}\right.$$

$$\left.+\frac{a}{1-a}+\frac{aq}{1-aq}+\cdots+\frac{aq^{n-1}}{1-aq^{n-1}}\right]=\sum_{n=1}^{\infty}\frac{n^2q^n}{1-q^n}, \tag{3.5}$$

which reduces to Bell's identity when $a=0$.

Bell pointed out that if in (3.5) one expands in powers of q and equates coefficients of like powers of q, one gets a curious polynomial in a, which must vanish identically. Expressing this identically vanishing polynomial arithmetically, one can get theorems on numbers of representations and also interesting recurrence formulae for functions of divisors. By giving 'a' special values in the polynomial, for example $a=-1$ or $a=\rho$, where ρ is a primitive rth root of unity, one obtains further theorems relating to restricted representations.

In view of Bell's remarks on the number-theoretic interest of (3.5), Bailey [13] gave elementary algebraic proof for it. Being quite instructive we give below in some details this proof.

Denote the left hand side of (3.5) by $F(a)$. Then

$$F(a)=\sum_{n=1}^{\infty}\left[\frac{q^n}{(1-q^n)^2}-\frac{aq^n}{(1-aq^n)^2}\right]\left[\frac{1}{1-q}+\frac{1}{1-q^2}+\cdots+\frac{1}{1-q^n}+\right.$$

$$\left.+\frac{a}{1-a}+\frac{aq}{1-aq}+\cdots+\frac{aq^{n-1}}{1-aq^{n-1}}\right],$$

and, on simplification, it can be shown that

$$F(a)-F(aq)=0.$$

Thus

$$F(a)=F(aq)=\cdots=F(aq^n).$$

Making $n\to\infty$, we get

$$F(a)=F(0)=\sum_{a=1}^{\infty}\frac{n^2q^n}{1-q^n},$$

Bell's identity being used in the final step.

In a letter to Bailey, Bell later mentioned still another identity, namely

$$8\sum_{\substack{m>1\\ m\,\mathrm{odd}}}\frac{q^m(1+q^{2m})}{(1-q^{2m})^2}\left[\frac{1}{1-q^2}+\frac{1}{1-q^4}+\cdots+\frac{1}{1-q^{2m-2}}\right]$$

$$=\sum_{m\,\mathrm{odd}}\left[\frac{(m^2-1)q^m}{1-q^{2m}}+\frac{4(m-1)q^m(1+q^{2m})}{(1-q^{2m})^2}\right]. \tag{3.6}$$

He was led to this identity from the theorem that, if m is odd, then the number of representations of m in the form

$$m = 2wx + xy + yz + zu + ux,$$

where $w, x, y, z > 0$, $u \geqslant 0$, is

$$\frac{1}{8}[\xi_2(m) - (4m+1)\xi_0(m) + 4\xi_1(m)],$$

$\xi_r(n)$ being the sum of the rth powers of all divisors of n. Bailey proved that (3.6) can be easily deduced from (3.5). Put $a = -1$ in (3.5) to obtain

$$2\sum_{n=1}^{\infty} \frac{q^n(1+q^{2n})}{(1-q^{2n})^2}\left[-\frac{1}{2} + \frac{1}{1-q^n} + \frac{1+q^2}{1-q^2} + \frac{1+q^4}{1-q^4} + \cdots + \frac{1+q^{2n-2}}{1-q^{2n-2}}\right]$$
$$= \sum_{n=1}^{\infty} \frac{n^2 q^n}{1-q^n},$$

Changing the sign of q and subtracting, we find that

$$4\sum_{n\,\mathrm{odd}} \frac{q^n(1+q^{2n})}{(1-q^{2n})^2}\left[-\frac{1}{2} + \frac{1}{1-q^{2n}} + \frac{1+q^2}{1-q^2} + \frac{1+q^4}{1-q^4} + \cdots + \frac{1+q^{2n-2}}{1-q^{2n-2}}\right]$$
$$= \sum_{n\,\mathrm{odd}} \frac{2n^2 q^n}{1-q^{2n}},$$

and so

$$8\sum_{n>1,\,\mathrm{odd}} \frac{q^n(1+q^{2n})}{(1-q^{2n})^2}\left[\frac{1}{1-q^2} + \frac{1}{1-q^4} + \cdots + \frac{1}{1-q^{2n-2}}\right]$$
$$= \sum_{n\,\mathrm{odd}} \frac{2n^2 q^n}{1-q^{2n}} + 4\sum_{n\,\mathrm{odd}} \frac{q^n(1+q^{2n})}{(1-q^{2n})^2}\left[\frac{1}{2} - \frac{1}{1-q^{2n}} + n - 1\right]$$
$$= \sum_{n\,\mathrm{odd}}\left[\frac{2(n^2-1)q^n}{1-q^{2n}} + \frac{4(n-1)q^n(1+q^{2n})}{(1-q^{2n})^2} - \frac{8q^{3n}}{(1-q^{2n})^3}\right].$$

Since

$$\sum_{n\,\mathrm{odd}} \frac{8q^{3n}}{(1-q^{2n})^3} = 4\sum_{n\,\mathrm{odd}}\sum_{m=1}^{\infty} m(m+1)q^{m(2m+1)}$$
$$= 4\sum_{m=1}^{\infty} \frac{m(m+1)q^{2m+1}}{1-q^{2(2m+1)}} = \sum_{n\,\mathrm{odd}} \frac{(n^2-1)q^n}{1-q^{2n}},$$

(3.6) follows.

It may be further noted, as a consequence of the above identities, that

$$\sum_{n=1}^{\infty} \xi_2(n)q^n = \sum_{1}^{\infty} \frac{n^2 q^n}{1-q^n},$$

$$\sum_{n=1}^{\infty} \xi_0(n)q^n = \sum_{1}^{\infty} \frac{q^n}{1-q^n}$$

$$\sum_{n=1}^{\infty} \xi_1(n)q^n = \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2},$$

$$\sum_{n=1}^{\infty} n\xi_0(n)q^n = \sum_{n=1}^{\infty} \frac{nq^n}{(1-q^n)^2}.$$

The above illustrations are enough to show how elegantly the Lambert series are related to various arithmetical functions.

4. Lambert series and Weierstrass's elliptic functions

We now pass on to exhibit the relationship of the Lambert series with the classical Weierstrass elliptic function.

Ramanujan [21] gave two formulae

$$\sum_{n=0}^{\infty} p(5n+4)x^n = 5\prod_{n=1}^{\infty} \frac{(1-x^{5n})^5}{(1-x^n)^6}, \tag{4.1}$$

$$\prod_{n=1}^{\infty} \frac{(1-x^n)^5}{1-x^{5n}} = 1 - 5\left(\frac{x}{1-x} - \frac{2x^2}{1-x^2} - \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \frac{6x^6}{1-x^6} - \cdots\right), \tag{4.2}$$

where $p(n)$ is the number of partitions of n. In addition Ramanujan stated the formula

$$x\prod_{n=1}^{\infty} \frac{(1-x^{5n})^5}{1-x^n} = \frac{x}{(1-x)^2} - \frac{x^2}{(1-x^2)^2} - \frac{x^3}{(1-x^3)^2} + \frac{x^4}{(1-x^4)^2} + \frac{x^6}{(1-x^6)^2} - \cdots, \tag{4.3}$$

and used it to prove (4.1). Bailey [15] showed that the proofs of (4.2) and (4.3) depend only on well-known formulae in elliptic functions which we, today, believe would be familiar to Ramanujan.

It is well known that

$$\mathscr{P}(u) = -\frac{\eta_1}{\omega_1} + \left(\frac{\pi}{2\omega_1}\right)^2 \operatorname{cosec}^2 \frac{\pi u}{2\omega_1} - 2\left(\frac{\pi}{\omega_1}\right)^2 \sum_{1}^{\infty} \frac{sq^{2s}}{1-q^{2s}} \cos\frac{s\pi u}{\omega_1},$$

where $\mathscr{P}(u)$ is the Weierstrass's elliptic function.

It is also easy to see that

$$\begin{aligned}\sum_{-\infty}^{\infty}{}' \frac{aq^n}{(1-aq^n)^2} &= \sum_{n=1}^{\infty}\left[\frac{aq^n}{(1-aq^n)^2} + \frac{q^n/a}{(1-q^n/a)^2}\right] \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m(a^m + a^{-m})q^{mn} = \sum_{m=1}^{\infty} \frac{m(a^m+a^{-m})q^m}{1-q^m},\end{aligned} \tag{4.4}$$

where $|q| < |a| < 1/|q|$. If we write $a = \exp(\pi iu/\omega_1)$, we get

$$\sum_{-\infty}^{\infty} \frac{aq^{2n}}{(1-aq^{2n})^2} = \frac{\eta\omega_1}{\pi^2} - \frac{\omega_1^2}{\pi^2}\wp(u).$$

Thus, if $b = \exp(\pi iv/\omega_1)$, we have

$$\sum_{-\infty}^{\infty}\left[\frac{aq^{2n}}{(1-aq^{2n})^2} - \frac{bq^{2n}}{(1-bq^{2n})^2}\right] = \frac{\omega_1^2}{\pi^2}[\wp(v)-\wp(u)] = \frac{\omega_1^2}{\pi^2}\,\frac{\sigma(u-v)\sigma(u+v)}{\sigma^2(u)\sigma^2(v)}.$$

We therefore find, after a little reduction and changing q^2 into q, that

$$\sum_{-\infty}^{\infty}\left[\frac{aq^{n}}{(1-aq^{n})^2} - \frac{bq^{n}}{(1-bq^{n})^2}\right] = a\Pi\left[\begin{matrix} ab, q/ab, b/a, qa/b, q, q, q, q; \\ a, a, q/a, q/a, b, b, q/b, q/b \end{matrix}\right] \tag{4.5}$$

where

$$\Pi\left[\begin{matrix} a_1, a_2, \ldots, \\ b_1, b_2, \ldots \end{matrix}\right] = \prod_{n=0}^{\infty} \frac{(1-a_1q^n)(1-a_2q^n)\cdots}{(1-b_1q^n)(1-b_2q^n)\cdots}.$$

The formula (4.5) is thus equivalent to the well-known expression for $\wp(u) - \wp(v)$ in terms of σ-functions.

For the proofs of both (4.2) and (4.3) we take $b = a^2$ in (4.5). First take $a = x$, $b = x^2$, $q = x^5$, and we get

$$\sum_{-\infty}^{\infty}\left[\frac{x^{5n+1}}{(1-x^{5n+1})^2} - \frac{x^{5n+2}}{(1-x^{5n+2})^2}\right] = x\prod_{1}^{\infty}\frac{(1-x^{5n})^5}{1-x^n},$$

which is (4.3).

Next take $a = \omega$, $b = \omega^2$, where $\omega = \exp(2\pi i/5)$, and the product on the right of (4.5) becomes

$$\frac{\omega(1-\omega^3)(1-\omega)}{(1-\omega)^2(1-\omega^2)^2}\prod_{1}^{\infty}\frac{(1-q^n)^5}{1-q^{5n}}.$$

The left hand side of (4.5) is, by (4.4)

$$\frac{\omega}{(1-\omega)^2} - \frac{\omega^2}{(1-\omega^2)^2} + \sum_{m=1}^{\infty}\frac{mq^m}{1-q^m}(\omega^m + \omega^{4m} - \omega^{2m} - \omega^{3m})$$

$$= \frac{\omega(1-\omega^3)(1-\omega)}{(1-\omega)^2(1-\omega^2)^2} + \sum_{m=1}^{\infty}\frac{mq^m\omega^m(1-\omega^m)(1-\omega^{2m})}{1-q^m}.$$

If we denote $\omega(1-\omega)(1-\omega^2)$ by A, it is easily seen that

$$\omega^m(1-\omega^m)(1-\omega^{2m})$$

has the values $0, A, -A, -A, A$ when m has the forms $5n$, $5n+1$, $5n+2$, $5n+3$, $5n+4$. Thus

$$\prod_{1}^{\infty}\frac{(1-q^n)^5}{1-q^{5n}} = 1 + B\sum_{n=0}^{\infty}\left[\frac{(5n+1)q^{5n+1}}{1-q^{5n+1}} - \frac{(5n+2)q^{5n+2}}{1-q^{5n+2}} - \right.$$

$$\left. - \frac{(5n+3)q^{5n+3}}{1-q^{5n+3}} + \frac{(5n+4)q^{5n+4}}{1-q^{5n+4}}\right],$$

where

$$\begin{aligned} B &= -(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4) \\ &= -5. \end{aligned}$$

This completes the proof of (4.2). It is interesting to note that both (4.2) and (4.3) are merely particular cases of the formula expressing $\wp(u) - \wp(2u)$ in terms of σ-functions, when the $\wp$-functions are replaced by the corresponding Fourier series.

This singular illustration picked out of Ramanujan's own work, to my mind is sufficient to bring home the close relationship of the Lambert series with the elliptic function theory with which Ramanujan is believed to have been well-acquainted. Bailey's proof also suggests as to how Ramanujan, himself, might have derived (4.2) and (4.3). One could, of course, cite many more such examples (see § 7 ahead).

5. Lambert series and basic hypergeometric series

This aspect of study of the Lambert series and basic hypergeometric series is, probably, the most effective tool, in handling a large number of queer looking unproved identities of Ramanujan. Perhaps, Ramanujan must have gone through the elements of these functions, although, it is not likely, that he may have used them to discover some of his remarkable identities mentioned, without proof, and now deducible directly from the transformation theory of basic hypergeometric series [1, 2, 3, 5, 9].

In what follows we shall use the following notation:

If

$$|q| < 1, \quad (a; q^k)_n = (1-a)(1-aq^k)\cdots(1-aq^{k(n-1)})$$

$$(a; q^k)_0 = 1 \text{ and } (a; q^k)_\infty = \prod_0^\infty (1 - aq^{kn}),$$

(for $k = 1$, we simply write $(a)_n$), and define a generalised basic hypergeometric series for $|z| < 1$, $|q| < 1$

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, \ldots\ldots, a_{r+1}; z \\ b_1, \ldots\ldots, b_r \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1; q)_n, \ldots\ldots (a_{r+1}; q)_n z^n}{(q; q)_n (b_1; q)_n \ldots\ldots (b_r; q)_n}.$$

Also, a generalized bilateral basic hypergeometric series is defined as

$${}_r\Psi_r\left[\begin{matrix} a_1, \ldots\ldots, a_r; q^k; z \\ b_1, \ldots\ldots, b_r \end{matrix}\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1; q^k)_n, \ldots\ldots (a_r; q^k)_n z^n}{(b_1; q^k)_n \ldots\ldots (b_r; q^k)_n},$$

convergent for

$$\left|\frac{b_1 b_2 \cdots b_r}{a_1 a_2 \cdots a_r}\right| < |z| < 1.$$

To illustrate the elegance and effectiveness of basic hypergeometric methods in proving and generalising a variety of q-identities, compared to other methods, alternative proofs are given here through these functions, of some of the key identities already proved in § 3 and § 4 by different methods.

The formula (3.1) can also be deduced from the basic analogue of Gauss's theorem for the sum of the hypergeometric series with unit argument. This analogue is

$$\sum_{n=0}^{\infty}\frac{(b)_n(c)_n}{(q)_n(d)_n}\left(\frac{d}{bc}\right)^n=\prod_{n=0}^{\infty}\left[\frac{(1-dq^n/b)(1-dq^n/c)}{(1-dq^n)(1-dq^n/bc)}\right]. \tag{5.1}$$

Differentiating with respect to d, and dividing by $(1-b)\,(1-c)$, we find that

$$\begin{aligned}\sum_{n=1}^{\infty}&\frac{(bq)_{n-1}(cq)_{n-1}}{(q)_n(d)_n}\left(\frac{d}{bc}\right)^{n-1}\left[\frac{1}{1-d}+\frac{1}{1-qd}+\cdots\frac{1}{1-q^{n-1}d}\right]\\&=\prod_{n=0}^{\infty}\left[\frac{(1-dq^n/b)(1-dq^n/c)}{(1-dq^n)(1-dq^n/bc)}\right]\\&\quad\times\sum_{n=0}^{\infty}\frac{q^n(1-d^2q^{2n}/bc)}{(1-dq^n)(1-dq^n/b)(1-dq^n/c)(1-dq^n/bc)}.\end{aligned} \tag{5.2}$$

Setting b and c tend to unity (assuming that $|q|<1$, $|d|<1$), we obtain

$$\sum_{n=1}^{\infty}\frac{(q)_{n-1}d^n}{(1-q^n)(d)_n}\left[\frac{1}{1-d}+\frac{1}{1-qd}+\cdots\frac{1}{1-q^{n-1}d}\right]=\sum_{n=0}^{\infty}\frac{dq^n(1+dq^n)}{(1-dq^n)^3}. \tag{5.3}$$

When $d=q$, this gives

$$\begin{aligned}\sum_{n=1}^{\infty}&\frac{q^n}{(1-q^n)^2}\left[\frac{1}{1-q}+\frac{1}{1-q^2}+\cdots+\frac{1}{1-q^n}\right]=\sum_{n=1}^{\infty}\frac{q^n(1+q^n)}{(1-q^n)^3}\\&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}m^2q^{mn}=\sum_{m=1}^{\infty}\frac{m^2q^m}{1-q^m},\end{aligned}$$

which is Bell's identity (3.1).

For general values of d, (5.3) similarly gives (3.4). It is evident that further results can be obtained, in a manner similar to the above from other formulae which sum basic series.

Thus, from the formula

$$\begin{aligned}&{}_6\phi_5\left[\begin{matrix}a, & q\sqrt{a}, & -q\sqrt{a}, & d, & e, & f;aq/def\\ & \sqrt{a}, & -\sqrt{a}, & aq/d, & aq/e, & aq/f\end{matrix}\right]\\&=\prod_{n=1}^{\infty}\left[\frac{(1-aq^n)(1-aq^n/ef)(1-aq^n/df)(1-aq^n/de)}{(1-aq^n/d)(1-aq^n/e)(1-aq^n/f)(1-aq^n/def)}\right]\end{aligned} \tag{5.4}$$

we can deduce, among other results, the formula

$$\begin{aligned}(1-a)\sum_{n=1}^{\infty}&\frac{(1-aq^{2n})q^n}{(1-q^n)^2(1-aq^n)^2}\left[\frac{1}{1-q}+\frac{1}{1-q^2}+\cdots+\frac{1}{1-q^n}\right.\\&\left.+\frac{a}{1-a}+\frac{aq}{1-aq}+\cdots+\frac{aq^{n-1}}{1-aq^{n-1}}\right]=\sum_{n=1}^{\infty}\frac{n^2q^n}{1-q^n}.\end{aligned} \tag{5.5}$$

For $a=0$ this again reduces to Bell's identity. For $a=q$ and $a=-q$, we obtain the formulae

$$(1-q)\sum_{n=1}^{\infty}\frac{(1-q^{2n+1})q^n}{(1-q^n)^2(1-q^{n+1})^2}\left[\frac{1+q}{1-q}+\frac{1+q^2}{1-q^2}+\cdots+\frac{1+q^n}{1-q^n}\right]$$

$$=\sum_{n=1}^{\infty}\frac{n^2q^n}{1-q^n}. \tag{5.6}$$

$$(1+q)\sum_{n=1}^{\infty}\frac{(1+q^{2n+1})q^n}{(1-q^n)^2(1+q^{n+1})^2}\left[\frac{1+q^2}{1-q^2}+\frac{1+q^4}{1-q^4}+\cdots+\frac{1+q^{2n}}{1-q^{2n}}\right]$$

$$=\sum_{n=1}^{\infty}\frac{n^2q^n}{1-q^n}. \tag{5.7}$$

In 1952, Bailey [14] showed that even (4.3), follows easily from the known sum of a well-poised basic bilateral hypergeometric series. The right hand side of (4.3) is

$$\sum_{n=0}^{\infty}\left[\frac{x^{5n+1}}{(1-x^{5n+1})^2}-\frac{x^{5n+2}}{(1-x^{5n+2})^2}-\frac{x^{5n+3}}{(1-x^{5n+3})^2}+\frac{x^{5n+4}}{(1-x^{5n+4})^2}\right]$$

$$=\sum_{n=-\infty}^{\infty}\left[\frac{x^{5n+1}}{(1-x^{5n+1})^2}-\frac{x^{5n+3}}{(1-x^{5n+3})^2}\right]$$

$$=\sum_{n=-\infty}^{\infty}\frac{x^{5n+1}(1-x^2)(1+x^{5n+2})(1-x^{5n+2})}{(1-x^{5n+1})^2(1-x^{5n+3})^2}$$

$$=\frac{x(1-x^2)(1-x^4)}{(1-x)^2(1-x^3)^2}\,{}_6\Psi_6\left[\begin{matrix}x^7, & -x^7, & x, & x, & x^3, & x^3;\\ x^2, & -x^2, & x^8, & x^8, & x^6, & x^6\end{matrix}\;x^5\right]$$

$$=\frac{x(1-x^2)(1-x^4)}{(1-x)^2(1-x^3)^2}\times$$

$$\times\prod_{n=1}^{\infty}\left[\frac{(1-x^{5n+4})(1-x^{5n+2})(1-x^{5n})^4(1-x^{5n-2})(1-x^{5n-4})}{(1-x^{5n-1})^2(1-x^{5n-3})^2(1-x^{5n+3})^2(1-x^{5n+1})^2}\right]$$

$$=x\prod_{n=1}^{\infty}\frac{(1-x^{5n})^5}{1-x^n},$$

where $q=x^5$ in the bilateral series. This completes the proof.

The other formula (4.2), namely

$$\frac{[(1-x)(1-x^2)(1-x^3)\ldots]^5}{(1-x^5)(1-x^{10})(1-x^{15})\ldots}$$

$$=1-5\left(\frac{x}{1-x}-\frac{2x^2}{1-x^2}-\frac{3x^3}{1-x^3}+\frac{4x^4}{1-x^4}+\frac{6x^6}{1-x^6}-\cdots\right), \tag{5.8}$$

which was stated by Ramanujan in one of his unpublished manuscript, but was also given elsewhere, can be expressed in the form that, if

$$f(x)=x^{1/5}\frac{(1-x)(1-x^4)(1-x^6)(1-x^9)\dots}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)\dots},$$

$$\frac{df}{fdx}=\frac{1}{5x}\prod_{n=1}^{\infty}\frac{(1-x^n)^5}{1-x^{5n}}.$$

A very complicated proof of this result was given by Darling whereas a short but more sophisticated proof was given by Mordell. Bailey [14] showed that (5.8) can also be deduced from the same sum of a bilateral series. In fact, from this sum we have

$$ {}_6\Psi_6\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, \omega\sqrt{a}, \omega^2\sqrt{q}, \omega^3\sqrt{a}, \omega^4\sqrt{a}; \\ \sqrt{a}, -\sqrt{a}, \omega^4 q\sqrt{a}, \omega^3 q\sqrt{a}, \omega^2 q\sqrt{a}, \omega q\sqrt{a} \end{matrix}\, q\right]$$

$$=\prod_{n=1}^{\infty}\frac{(1-q^n)(1-aq^n)(1-q^n/a)(1-q^{5n})(1-q^n\sqrt{a})(1-q^n/\sqrt{a})}{(1-q^{5n}a^{5/2})(1-q^{5n}/a^{5/2})},$$

where $\omega=\exp(2\pi i/5)$. Now, if $a\to 1$, the product on the right becomes

$$\prod_{n=1}^{\infty}\frac{(1-q^n)^5}{1-q^{5n}}.$$

The series on the left is

$$1+\frac{1-a^{5/2}}{(1-a)(1-a^{1/2})}\sum_{n=1}^{\infty}\left[\frac{(1-aq^{2n})(1-a^{1/2}q^n)q^n}{1-a^{5/2}q^{5n}}-\frac{(q^{2n}-a)(q^n-a^{1/2})q^n}{a^{5/2}-q^{5n}}\right]. \tag{5.9}$$

When $a\to 1$, (5.9) becomes

$$1-5\sum_{n=1}^{\infty}\left[\frac{q^n-2q^{2n}-3q^{3n}+4q^{4n}+4q^{6n}-3q^{7n}-2q^{8n}+q^{9n}}{(1-q^{5n})^2}\right].$$

Now

$$\sum_{n=1}^{\infty}\frac{q^{an}}{(1-q^{5n})^2}=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(m+1)q^{(5m+a)n}=\sum_{m=0}^{\infty}\frac{(m+1)q^{5m+a}}{1-q^{5m+a}},$$

and so we have

$$\begin{aligned}\prod_{n=1}^{\infty}\frac{(1-q^n)^5}{1-q^{5n}}=1-5\sum_{m=0}^{\infty}\Bigg[&\frac{(m+1)q^{5m+1}}{1-q^{5m+1}}-\frac{2(m+1)q^{5m+2}}{1-q^{5m+2}}-\\ &-\frac{3(m+1)q^{5m+3}}{1-q^{5m+3}}+\frac{4(m+1)q^{5m+4}}{1-q^{5m+4}}+\frac{4mq^{5m+1}}{1-q^{5m+1}}-\\ &-\frac{3mq^{5m+2}}{1-q^{5m+2}}-\frac{2mq^{5m+3}}{1-q^{5m+3}}+\frac{mq^{5m+4}}{1-q^{5m+4}}\Bigg]\\ =1-5\sum_{m=0}^{\infty}\Bigg[&\frac{(5m+1)q^{5m+1}}{1-q^{5m+1}}-\frac{(5m+2)q^{5m+2}}{1-q^{5m+2}}-\\ &-\frac{(5m+3)q^{5m+3}}{1-q^{5m+3}}+\frac{(5m+4)q^{5m+4}}{1-q^{5m+4}}\Bigg].\end{aligned}$$

This completes the proof of (4.2).

The above applications provide a *prima facie* evidence of the elegance with which the basic hypergeometric series may be utilized in the derivation and generalisation of a variety of q-identities involving Lambert series.

6. Ramanujan and a generalised Lambert series

We now turn to numerous identities given by Ramanujan in which Lambert series and its generalisations find a mention. A series of the form

$$\sum_{n=-\infty}^{\infty} (-1)^{\varepsilon n} q^{\lambda n^2} R(q^n), \tag{6.1}$$

where $\varepsilon = 0$ or 1, $\lambda > 0$, and $R(x)$ is a rational function of x, will be called a generalised Lambert series.

In the 'Lost' Notebook [7] Ramanujan gave a number of identities involving series of the type (6.1). I mention below eleven of them, namely

$$[C(q)]^{-3} = \frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{5n+2}}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{n}}{1-q^{5n+1}}} \tag{6.2}$$

$$= \frac{\displaystyle\sum_{n=0}^{\infty} q^{5n^2+4n}\frac{1+q^{5n+2}}{1-q^{5n+2}} - \sum_{n=0}^{\infty} q^{5n^2+6n+1}\frac{1+q^{5n+3}}{1-q^{5n+3}}}{\displaystyle\sum_{n=0}^{\infty} q^{5n^2+2n}\frac{1+q^{5n+1}}{1-q^{5n+1}} - \sum_{n=0}^{\infty} q^{5n^2+8n+3}\frac{1+q^{5n+4}}{1-q^{5n+4}}}. \tag{6.3}$$

$$(q^5;q^5)_\infty^2 G(q) = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+2}} \tag{6.4}$$

$$(q^5;q^5)_\infty^2 H(q) = \sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1-q^{5n+1}} \tag{6.5}$$

$$(q^5;q^5)_\infty^2 \frac{G(q)^2}{H(q)} = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+1}} \tag{6.6}$$

$$(q^5;q^5)_\infty^2 \frac{H(q)^2}{G(q)} = \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{5n+2}} \tag{6.7}$$

$$C(q) = \frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{n}}{1-q^{5n+2}}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1-q^{5n+1}}} \tag{6.8}$$

$$C(q)^2 = \frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{n}}{1-q^{5n+1}}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1-q^{5n+1}}} \tag{6.9}$$

$$C(q)^2 = \frac{\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+2}}}{\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{5n+2}}}, \tag{6.10}$$

$$(q^5;q^5)_\infty^2 G(q) = \sum_{n=-\infty}^{\infty} \frac{q^{4n}}{1-q^{10n+1}} \tag{6.11}$$

$$(q^5;q^5)_\infty^2 H(q) = \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{10n+3}}, \tag{6.12}$$

where the Rogers-Ramanujan functions

$$G(q) = [(q;q^5)_\infty (q^4;q^5)_\infty]^{-1}$$

$$H(q) = [(q^2;q^5)_\infty (q^3;q^5)_\infty]^{-1}$$

and $C(q)$ is the Ramanujan's continued function with its value equal to $C(q) = G(q)/H(q)$.

To prove the equivalences (6.2–6.12) let us consider the famous Ramanujan's sum of a bilateral basic hypergeometric series ${}_1\Psi_1$, namely

$${}_1\Psi_1(a;b;q;z) = \sum_{-\infty}^{\infty} [a]_n z^n/[b]_n$$

$$= \frac{[b/a;q]_\infty [az;q]_\infty [q/az;q]_\infty [q;q]_\infty}{[q/a;q]_\infty [b/az;q]_\infty [b;q]_\infty [z;q]_\infty}. \tag{6.13}$$

Hardy [17] had called this sum as a remarkable formula with many parameters. It is noteworthy that more than half-a-dozen different proofs of this interesting sum have been given by different mathematicians from time to time.

Thus,

$$\frac{1}{(1-\alpha)} {}_1\Psi_1(\alpha;\alpha q^\mu;q^\mu;z); \quad (|q^\mu| < |\alpha|, |z| < 1) \tag{6.14}$$

$$= \sum_{n=-\infty}^{\infty} \frac{z^n}{1-\alpha q^{\mu n}}$$

$$= \frac{[q^\mu;q^\mu]_\infty^2 [z\alpha;q^\mu]_\infty [q^\mu/\alpha z;q^\mu]_\infty}{[q^\mu/\alpha;q^\mu]_\infty [q^\mu/z;q^\mu]_\infty [\alpha;q^\mu]_\infty [z;q^\mu]_\infty}. \tag{6.15}$$

We now proceed to prove that

$$\sum_{n=-\infty}^{\infty} (\alpha z)^n q^{\mu n^2} \frac{(1-z\alpha q^{2\mu n})}{(1-\alpha q^{\mu n})(1-zq^{\mu n})}$$

$$= \sum_{-\infty}^{\infty} \frac{z^n}{(1-\alpha q^{\mu n})}; \quad |\alpha| < 1, |z| < 1. \tag{6.16}$$

We have

$$\sum_{n=0}^{\infty} \frac{z^n}{1-\alpha q^{\mu n}} = \sum_{n=0}^{\infty} z^n \sum_{m=0}^{\infty} q^{m\mu n} \alpha^m$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} z^{n+m} q^{m\mu(m+n)} \alpha^m + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} z^n \alpha^{m+n+1} q^{n\mu(m+n+1)}$$

$$= \sum_{m=0}^{\infty} z^m \alpha^m q^{\mu m^2} \sum_{n=0}^{\infty} z^n q^{nm\mu} + \sum_{n=0}^{\infty} z^n \alpha^{n+1} q^{n\mu(n+1)} \sum_{m=0}^{\infty} q^{mn\mu} \alpha^m$$

$$= \sum_{m=0}^{\infty} \frac{(z\alpha)^m q^{\mu m^2}}{1-zq^{m\mu}} + \alpha \sum_{n=0}^{\infty} \frac{(\alpha z)^n q^{n^2\mu+n\mu}}{1-\alpha q^{n\mu}}.$$

$$\therefore \sum_{n=0}^{\infty} \frac{z^n}{1-\alpha q^{\mu n}} = \sum_{n=0}^{\infty} \frac{(\alpha z)^n q^{\mu n^2}(1-z\alpha q^{2n\mu})}{(1-zq^{n\mu})(1-\alpha q^{n\mu})}. \tag{6.17}$$

Using (6.17), twice, we get (6.16).

Thus, we get by using (6.14) and (6.15) in (6.16), the main result in the equivalent form ($|q^\mu| < |z|, |\alpha| < 1$)

$$\sum_{-\infty}^{\infty} q^{\mu n^2} \frac{(1-\alpha z q^{2\mu n})(\alpha z)^n}{(1-\alpha q^{\mu n})(1-zq^{\mu n})}$$

$$= \frac{[q^\mu; q^\mu]_\infty^2 [\alpha z; q^\mu]_\infty [q^\mu/\alpha z; q^\mu]_\infty}{[q^\mu/\alpha; q^\mu]_\infty [q^\mu/z; q^\mu]_\infty [\alpha; q^\mu]_\infty [z; q^\mu]_\infty}. \tag{6.18}$$

The cases $\alpha = q^i, i = 1, 2, 3, 4; z = q^j, 0 < j < 4; \mu = 5$ were considered by Andrews [8].

It may be further remarked that the results of Andrews can also follow directly, as very special cases of the Bailey's sum of a well-poised ${}_6\Psi_6$-series.

Following Andrews [8], to prove (6.2) and (6.3), take in the main result (6.18), $\mu = 5$, $z = \alpha = q^2$, to get the numerator of (6.2) and $\mu = 5$, $z = \alpha = q$, to get the denominator of (6.2). Dividing the two expressions and using the product equivalent for $C(q)$, one gets (6.3).

In (6.15) take

(a) $\mu = 5$, $\alpha = q^2$, $z = q$, to obtain (6.4).
(b) $\mu = 5$, $\alpha = q$, $z = q^3$, to obtain (6.5).
(c) $\mu = 5$, $\alpha = q$, $z = q$, to obtain (6.6).
(d) $\mu = 5$, $\alpha = q^2$, $z = q^2$, to obtain (6.7).

To obtain

(e) (6.8) use (6.4) and (6.5)
(f) (6.9) use (6.5) and (6.6)
(g) (6.10) use (6.4) and (6.7).

To obtain (6.11), in (6.15) let $q \to q^2$, and take $\mu = 5$, $\alpha = q$, $z = q^4$, and finally, to obtain (6.12), in (6.15) let $q \to q^2$ and take $\mu = 5$, $\alpha = q^3$, $z = q^2$. In (6.15) if we take

$\mu=4$, $z=\alpha=q$ and also $z=q^2$, $\alpha=q$ the resulting expressions give

$$\frac{\sum_{-\infty}^{\infty}\dfrac{q^n}{1-q^{4n+1}}}{\sum_{-\infty}^{\infty}\dfrac{q^{2n}}{1-q^{4n+1}}}=\prod_{n\geqslant 0}\frac{(1-q^{4n+2})^4}{(1-q^{4n+1})^2(1-q^{4n+3})^2}$$
$$=\left[1+\frac{q}{1+}\,\frac{q+q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^2+q^4}{1+}\,\frac{q^5}{1+}\cdots\right]^2,$$

which gives an expression for the square of another continued fraction identity of Ramanujan found in the 'Lost' Notebook and does not seem to have been mentioned earlier.

Again, in (6.15), taking $\mu=6, \alpha=q^2, z=q$ and also $\mu=6, \alpha=q^2, z=q^2$, the resulting expressions give

$$\sum_{-\infty}^{\infty}\frac{q^n}{1-q^{6n+2}}\Big/\sum_{-\infty}^{\infty}\frac{q^{2n}}{1-q^{6n+2}}=\prod_{n\geqslant 0}\frac{(1-q^{6n+3})^2}{(1-q^{6n+1})(1-q^{6n+5})}$$
$$=1+\frac{q+q^2}{1+}\,\frac{q^2+q^4}{1+}\,\frac{q^3+q^6}{1+}\cdots,$$

a new representation for another known continued fraction due to Watson [29], Gordon [11] and Andrews [8].

Similarly, in (6.15) taking $\mu=8$, $\alpha=q^5$, $z=q$ and also $\mu=8$, $\alpha=q^3$, $z=q^3$, we get on taking the quotient of the resulting identities an expression representing another continued fraction

$$1+\frac{q+q^2}{1+}\,\frac{q^4}{1+}\,\frac{q^3+q^6}{1+}\,\frac{q^8}{1+}\cdots,$$ given in the 'Lost' note book.

Denis [16], in 1988 used (6.15) and (6.16) in the form

$$\sum_{n=-\infty}^{\infty}\frac{q^{in}}{1-q^{\mu n+j}}=\frac{(q^\mu;q^\mu)_\infty^2(q^{i+j};q^\mu)_\infty(q^{\mu-i-j};q^\mu)_\infty}{(q^j;q^\mu)_\infty(q^{\mu-j};q^\mu)_\infty(q^i;q^\mu)_\infty(q^{\mu-i};q^\mu)_\infty} \tag{6.19}$$

and

$$\sum_{n=-\infty}^{\infty}\frac{q^{in}}{1-q^{\mu n+j}}=\sum_{n=-\infty}^{\infty}q^{\mu n^2+(i+j)n}\frac{(1-q^{2\mu n+i+j})}{(1-q^{\mu n+i})(1-q^{\mu n+j})} \tag{6.20}$$

and listed some very interesting new results of different types deducible from them.

To mention a few we have

$$\sum_{n=-\infty}^{\infty}\frac{q^{5n^2+3n}(1-q^{10n+3})}{(1-q^{5n+1})(1-q^{5n+2})}=(q^5;q^5)_\infty^2\,G(q)$$
$$(\mu=5, i=1, j=2), \tag{6.21}$$

$$\sum_{n=-\infty}^{\infty}\frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+1})(1-q^{5n+3})}=(q^5;q^5)_\infty^2\,H(q)$$
$$(\mu=5, i=1, j=3), \tag{6.22}$$

$$\sum_{n=-\infty}^{\infty}\frac{q^{5n^2+2n}(1+q^{5n+1})}{(1-q^{5n+1})}=(q^5;q^5)^2_\infty G(q)^2/H(q)$$
$$(\mu=5, i=1, j=1), \qquad (6.23)$$

vhich is $(3.11)_R$ of Andrews [8],

$$\sum_{n=-\infty}^{\infty}q^{5n^2+4n}\frac{(1+q^{5n+2})}{(1-q^{5n+2})}=[q^5;q^5]^2_\infty H(q)^2/G(q)$$
$$(\mu=5, i=2, j=2) \qquad (6.24)$$

vhich is $(3.12)_R$ of Andrews [8],

$$\sum_{n=-\infty}^{\infty}\frac{q^{10n^2+5n}(1-q^{20n+5})}{(1-q^{10n+2})(1-q^{10n+3})}=(q^5;q^5)^2_\infty H(q)$$
$$(\mu=10, i=2, j=3), \qquad (6.25)$$

$$\sum_{n=-\infty}^{\infty}\frac{q^{10n^2+5n}(1-q^{20n+5})}{(1-q^{10n+4})(1-q^{10n+1})}=(q^5;q^5)^2_\infty G(q)$$
$$(\mu=10, i=4, j=1), \qquad (6.26)$$

$$\sum_{n=-\infty}^{\infty}\frac{q^{10n^2+7n}(1-q^{20n+7})}{(1-q^{10n+5})(1-q^{10n+2})}=\sum_{n=-\infty}^{\infty}\frac{q^{5n}}{1-q^{10n+2}}$$
$$=[-q^5;q^5]^2_\infty[q^7;q^{10}]_\infty[q^3;q^{10}]_\infty G(q^2)$$
$$(\mu=10, i=5, j=2) \qquad (6.27)$$

$$\sum_{n=-\infty}^{\infty}\frac{q^{10n^2+7n}(1-q^{20n+7})}{(1-q^{10n+3})(1-q^{10n+4})}=\sum_{n=-\infty}^{\infty}\frac{q^{3n}}{1-q^{10n+4}}$$
$$=[q^{10};q^{10}]^2_\infty H(q^2)$$
$$(\mu=10, i=3, j=4). \qquad (6.28)$$

$$\sum_{-\infty}^{\infty}\frac{q^{7n^2+5n}(1-q^{14n+5})}{(1-q^{7n+2})(1-q^{7n+3})}=\sum_{-\infty}^{\infty}\frac{q^{2n}}{1-q^{7n+3}}$$
$$=[q^7;q^7]^2_\infty/[q^4;q^7]_\infty[q^3;q^7]_\infty$$
$$(\mu=7, i=2, j=3) \qquad (6.29)$$

$$\sum_{-\infty}^{\infty}\frac{q^{7n^2+4n}(1-q^{14n+4})}{(1-q^{7n+1})(1-q^{7n+3})}=\sum_{-\infty}^{\infty}\frac{q^{n}}{1-q^{7n+3}}$$
$$=[q^7;q^7]^2_\infty/[q;q^7]_\infty[q^6;q^7]_\infty$$
$$(\mu=7, i=1, j=3) \qquad (6.30)$$

$$\sum_{-\infty}^{\infty}q^{7n^2+6n}\frac{(1+q^{7n+3})}{(1-q^{7n+3})}=\sum_{-\infty}^{\infty}\frac{q^{3n}}{1-q^{7n+3}}$$
$$=\frac{[q^7;q^7]^2_\infty[q^6;q^7]_\infty[q;q^7]_\infty}{[q^3;q^7]^2_\infty[q^4;q^7]^2_\infty}$$
$$(\mu=7, i=1, j=3) \qquad (6.31)$$

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{10n^2+7n}(1-q^{20n+7})}{(1-q^{10n+5})(1-q^{10n+2})}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{10n^2+7n}(1-q^{20n+7})}{(1-q^{10n+3})(1-q^{10n+4})}} = \frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{5n}}{(1-q^{10n+2})}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1-q^{10n+4}}}$$

$$= \frac{[q^3;q^{10}]_\infty [q^7;q^{10}]_\infty}{(q^5;q^5)_\infty} C(q^2) \tag{6.32}$$

ollows from (6.27) and (6.28)).

A different type of double series representations can be obtained if, for example, ne considers the following identity of Verma and Jain [28] for Rogers–Ramanujan ype of identities related to modulus 57:

$$(-q;q)_\infty \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(q^6;q^6)_{n+2r} q^{2n^2+12nr+24r^2+6n+24r}}{(q^6;q^6)_r (q^2;q^2)_n (q^2;q^2)_{2n+6r+2} (-q^3;q^3)_{2r+1}}$$

$$= (q^{57};q^{57})_\infty (q^{54};q^{57})_\infty (q^3;q^{57})_\infty / (q;q)_\infty .$$

If we use (6.15) for $\mu = 57$, $\alpha = q^{27} = z$ and equate the products in above identity, e get the interesting representation

$$\left[\frac{[(q^{27};q^{57})_\infty (q^{30};q^{57})_\infty]^2}{[q^{57};q^{57}]_\infty} \right]^{-1} (q^2;q^2)_\infty \times$$

$$\times \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(q^6;q^6)_{n+2r} q^{2n^2+12nr+24r^2+6n+24r}}{(q^6;q^6)_r (q^2;q^2)_n (q^2;q^2)_{2n+6r+2} (-q^3;q^3)_{2r+1}}$$

$$= \sum_{-\infty}^{\infty} q^{57n^2+54n}(1+q^{57n+27})/(1-q^{57n+27}). \tag{6.33}$$

Similar identities, even, for quotients of two double series, as in (6.33), corresponding o other modulii, can be written down from various other results of Verma and Jain 28]. Also, if one scans the L J Slater's [23] list of identities one can give scores of ther representations on the above lines, for different modulii.

Very recently, in 1989, Andrews and Garvan [10] have proved the truth of certain dentities which are equivalent to the mock theta conjectures. In fact, they have onsidered two families of fifth order mock theta functions of Ramanujan and have hown that the identities in each family are equivalent (i.e. if one is true all five are rue and if one is false all five are false). They have used the method of generalised ambert series to prove them, and this enabled them to reduce all their computations o simple manipulations of these series. I sketch, below their method of proof, by educing only one of the ten Ramanujan's unproved identities, involving mock theta unctions of the fifth order.

Ramanujan had stated, without proof, that

$$M_1(q) \equiv \chi_0(q) - 2 - 3\Phi(q) + A(q) = 0$$

$$M_3(q) \equiv \varphi_0(-q) + \Phi(q) - (q^5;q^5)_\infty G(q^2)H(q)/H(q^2) = 0 \tag{6.34}$$

here χ_0, φ_0 are two of the ten fifth order mock theta functions and $\Phi(q)$, $A(q)$, $G(q)$

and $H(q)$ are certain auxilliary functions associated with them. (For definitions of all these functions one is referred to the original work of Andrews and Garvan [10]). They proceed to prove that

$$\begin{aligned}
&M_1(q^2)+3M_3(q^2)\\
&=\chi_0(q^2)+3\varphi_0(-q^2)-2+A(q^2)-\frac{3(q^{10};q^{10})_\infty G(q^4)H(q^2)}{H(q^4)}\\
&=\theta_4(0,-q)G(-q)+\theta_4(0,q)G(q)+A(q^2)-\\
&\quad -3(q^{10};q^{10})_\infty G(q^4)H(q^2)/H(q^4)\\
&=\frac{1}{(q^{10};q^{10})_\infty}\left[\sum_{-\infty}^{\infty}\frac{(-q^2)^n}{1-(-1)^n q^{5n+1}}+\sum_{-\infty}^{\infty}\frac{(-q^2)^n}{1+q^{5n+1}}+\right.\\
&\quad\left.+\sum_{-\infty}^{\infty}\frac{q^{2n}}{1-q^{10n+2}}-3\sum_{-\infty}^{\infty}\frac{(-q^2)^n}{1-q^{10n+2}}\right],
\end{aligned}\tag{6.35}$$

where we have replaced each term by its generalised Lambert series equivalent. It may further be remarked that each sum in (6.35) can be put by (6.19) equal to several ${}_1\Psi_1$-series each with base q^{10}. Andrews and Garvan proved that (6.35) equals

$$\frac{2}{(q^{10};q^{10})_\infty}\sum_{-\infty}^{\infty} q^{4n+2}/(1-q^{10n+6})\tag{6.36}$$

which is shown to be identically equal to zero (writing its equivalent infinite products). This proves the desired identity.

Since Ramanujan is believed to have an adequate background of the theory of elliptic functions and had also discovered the sum of a ${}_1\Psi_1$ series, it is reasonable to expect that he must have been conversant with the Lambert series and its bilateral extension.

The above method of proof, through Lambert series, given by Andrews and Garvan to establish the truth of Ramanujan's "unproven" identities, hence, is indicative of how Ramanujan, himself, might have arrived at the truth of his conjectures. Andrews and Garvan, thus, give credence to and help to unfold the mysticism popularly associated with Ramanujan's ingeniuty and creative thinking process.

It would, therefore, indeed be a marvellous achievement, if one could prove the numerous other "unproven" identities of Ramanujan also, in the spirit—"how Ramanujan might have proved them".

Nevertheless, the various other techniques (which might not have been known to Ramanujan) used, later, to prove his results have not only helped us to knit together the scores of his scattered identities in one net but also have helped to classify and generalise them to get a better insight in the ideas underlying their structure. However, these sophisticated methods, fall short in projecting the greatness of Ramanujan's genius, vision and intuitive power.

7. Certain other miscelleaneous identities and Lambert series

I next mention two recent applications of Lambert series, one each in sorting theory and Weierstrass's elliptic function theory.

Uchimura [25] in his recent works has considered the polynomial $U_m(x)$ which is defined as follows:

$$U_1(x) = x,$$
$$U_m(x) = mx^m + (1 - x^m)U_{m-1}(x) \quad (m \geqslant 2).$$

For instance:

$$U_1(x) = x,$$
$$U_2(x) = x + 2x^2 - x^3,$$
$$U_3(x) = x + 2x^2 + 2x^3 - x^4 - 2x^5 + x^6,$$
$$U_4(x) = x + 2x^2 + 2x^3 + 3x^4 - 3x^5 - x^6 - 2x^7 + x^8 + 2x^9 - x^{10}.$$

It is clear that $U_m(x)$ is a polynomial of degree $m(m+1)/2$.

The polynomial $U_m(x)$ has its origin in the analysis of the data structure called "heap" [24].

He proved the following two theorems.

Theorem 1. *Let*

$$U_m(x) = \sum_{n=1}^{m(m+1)/2} a_n^{(m)} x^n.$$

Then $a_n^{(m)} = d(n)$ for any $n \leqslant m$, where the divisor function $d(n)$ is the number of divisors of n, including 1 *and n.*

Theorem 2.

$$\sum_{n=1}^{\infty} nx^n \prod_{j=n+1}^{\infty} (1 - x^j)$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n(n+1)/2}}{(1-x)(1-x^2)\cdots(1-x^{n-1})(1-x^n)^2}$$
$$= \sum_{n=1}^{\infty} \frac{x^n}{1-x^n}. \tag{7.1}$$

Uchimura [26] later generalized (7.1) in the form that, for any positive integer m,

$$\sum_{k=1}^{n} \frac{(-1)^{k-1} q^{k(k+1)/2}}{1-q^{k+m-1}} \begin{bmatrix} n \\ k \end{bmatrix} = \sum_{k=1}^{n} \frac{q^k}{1-q^k} \Big/ \begin{bmatrix} k+m-1 \\ k \end{bmatrix}, \tag{7.2}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} q^{k(k+1)/2}}{(1-q)(1-q^2)\cdots(1-q^k)(1-q^{k+m-1})} = \sum_{k=1}^{\infty} \frac{q^k}{1-q^k} \Big/ \begin{bmatrix} k+m-1 \\ k \end{bmatrix}. \tag{7.3}$$

Putting $m = 1$ in (7.2) and (7.3), we get

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{(1-q^k)} q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} = \sum_{k=1}^{n} \frac{q^k}{1-q^k}, \tag{7.4}$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is the Gaussian polynomial defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q)\cdots(1-q^n)}{(1-q)\cdots(1-q^k)(1-q)\cdots(1-q^{n-k})}, \quad \text{if } 0 \leqslant k \leqslant n,$$
$$= 0, \quad \text{otherwise,}$$

and (7.1) respectively.

The above result (7.4) was proposed by L. Van Hamme [27] as an advanced problem.

Andrews [6] proved the identity (7.4) by using an identity of hypergeometric functions (7.2) is a finite version of (7.1).

As a final application we consider a result of Ramanujan [21] in connection with the development of obtaining formulae for $r_{2s}(n)$, the number of representation of n as sums of $2s$ squares. Ramanujan introduced the function $\phi_{r,s}(x) = \sum_m \sum_n m^r n^s \cdot x^{mn}$, symmetric in r and s and expressed $\phi_{r,s}(x)$, when $(r+s)$ is odd, as polynomials in the transcedentals in P, Q and R where

$$P = 1 - 24\phi_{0,1}(x) = 1 - 24 \sum_{n=1}^{\infty} \frac{nx^n}{1-x^n}$$

$$Q = 1 + 240\phi_{0,3}(x) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3x^n}{1-x^n}$$

$$R = 1 - 504\phi_{0,5}(x) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5x^n}{1-x^n}.$$

V Ramamani [22] in 1987 defined two allied functions, viz

$$\Psi_{r,s}(x) = \sum_m \sum_n (-1)^{n-1} m^r n^s x^{mn}$$

$$F_{r,s}(x) = \sum_m \sum_n (2m-1)^r n^s x^{(2m-1)n/2}, \quad r, s > 0$$

which are not symmetric in r and s and considered the problem of obtaining the polynomial expressions for them when $(r+s)$ is odd. She succeeded in expressing $\Psi_{r,s}(x)$ (when $(r+s)$ is odd) as polynomial in e_1, P_1 and Q_1, where $e_1 = \wp(\pi)$ (where $\wp$ is Weierstrassian elliptic function)

$$P_1 = 1 + 8 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}nx^n}{1-x^n} = 1 + 8\Psi_{0,1}(x)$$

$$Q_1 = 1 - 16 \sum_{1}^{\infty} \frac{(-1)^{n-1}n^3x^n}{1-x^n} = 1 - 16\Psi_{0,3}(x).$$

The behaviour of the function $F_{r,s}(x)$ is similar to that of $\Psi_{r,s}(x)$ and can be expressed, when $r+s$ is odd, as polynomials in e_3, P_2 and Q_2 where

$$e_3 = \wp(\pi\tau), \ \operatorname{Im} \tau > 0,$$

$$P_2 = -8 \sum_{m=1}^{\infty} \frac{mx^{m/2}}{1-x^m} = -8F_{0,1}(x)$$

$$Q_2 = 16 \sum_{m=1}^{\infty} \frac{m^3 x^{m/2}}{1-x^m} = 16F_{0,3}(x).$$

Further by noticing that $\Psi_{r,s}(x)$ and $F_{r,s}(x)$ are functions of $\phi_{r,s}(x)$, she expressed $\Psi_{r,s}(x)$ as polynomials in e_1, P_1 and Q_1 and $F_{r,s}(x)$ in e_3, P_2 and Q_2.

8. Lambert series and mock theta functions of order 'six'

Very recently, Andrews and Dean Hickerson [10a] proved five identities involving seven new mock theta functions found in the 'Lost' notebook. Andrews and Hickerson 'formally' labelled these functions as mock theta functions of order 'six'. The seven mock theta functions are defined as below;

$$\phi(q) = \sum_{n\geqslant 0} \frac{(-)^n q^{n^2}(q;q^2)_n}{(-q)_{2n}} \tag{8.1}$$

$$\Psi(q) = \sum_{n\geqslant 0} \frac{(-)^n q^{(n+1)^2}(q;q^2)_n}{(-q)_{2n+1}} \tag{8.2}$$

$$\rho(q) = \sum_{n\geqslant 0} \frac{q^{1/2n(n+1)}(-q)_n}{(q;q^2)_{n+1}} \tag{8.3}$$

$$\sigma(q) = \sum_{n\geqslant 0} \frac{q^{1/2n(n+2)}(-q)_n}{(q;q^2)_{n+1}} \tag{8.4}$$

$$\lambda(q) = \sum_{n\geqslant 0} \frac{(-)^n q^n(q;q^2)_n}{(-q)_n} \tag{8.5}$$

$$\mu(q) = \sum_{n\geqslant 0} \frac{(-)^n(q;q^2)_n}{(-q)_n} \tag{8.6}$$

$$\gamma(q) = \sum_{n\geqslant 0} \frac{q^{n^2}(q)_n}{(q^3;q^3)_n}. \tag{8.7}$$

It should be noted that (8.6) does not converge. However, we give it a meaning by taking the average of sequence of odd and sequence of even partial sums, each of which converges.

Andrews and Hickerson derived the following simple Lambert series representations for five of these functions:-

$$J_{1,3}\phi(q) = 2\sum_{r=-\infty}^{\infty} \frac{q^{r(3r+1)/2}}{1+q^{3r}} \tag{8.8}$$

$$J_{1,3}\Psi(q) = 2\sum_{r=-\infty}^{\infty} \frac{q^{(r+1)(3r+2)/2}}{1+q^{3r+1}} \tag{8.9}$$

$$J_{1,6}\rho(q) = \sum_{r=-\infty}^{\infty} \frac{(-)^r q^{r(3r+4)}}{1-q^{6r+1}} \tag{8.10}$$

$$J_{1,6}\sigma(q)=\sum_{r=-\infty}^{\infty}\frac{(-)^r q^{(r+1)(3r+1)}}{1-q^{6r+3}} \tag{8.11}$$

$$(q)_\infty\gamma(q)=3\sum_{r=-\infty}^{\infty}\frac{(-)^r q^{r(3r+1)/2}}{1+q^r+q^{2r}}, \tag{8.12}$$

where $J_{a,m}=(q^a,q^{m-a},q^m;q^m)_\infty$.

Similar representations for $\lambda(q)$ and $\mu(q)$ have also been given by them but are not so elegant (see [10a]; (4.19) and (4.20)).

They used these alternative representations to prove the various 'unproven' identities connecting them, as given by Ramanujan.

It may be remarked that (8.12) is also found in earlier works of N J Fine (for details see Agarwal ([3a]).

The rationality of the 'adhoc' labelling of these seven mock theta functions as of order 'six' has, very recently, aroused much interest. This has motivated Agarwal [3a] to give a formal definition of the order of a mock theta function, in an attempt to settle a seventy-year old 'adhocism' in naming such functions as mock theta functions of orders three, five, seven and six. Agarwal's discussion also leaves a question mark on the existence of 'mock theta functions' of orders other than those of orders three, five and seven given by Ramanujan.

I think I have said enough to justify the effectiveness and elegance with which Lambert series can be used to partially lift the curtain of ignorance regarding Ramanujan's creative and ever vibrant genius.

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Behaviour of an associated conjugate series of a Fourier series

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Abstract. In this paper we obtain a generalisation of a theorem of Mohanty and Ray (*Can. J. Math.* **21** (1969) 535–551)

Keywords. Absolute Cesàro summability; bounded variation; conjugate series; Fourier series.

1. Introduction

Let δ and β be real numbers and $\{A_n^{\delta,\beta}\}$ be a sequence of constants defined by the series

$$\frac{1}{(1-z)^{\delta+1}}\left(\log\frac{a}{1-z}\right)^{\beta} = \sum_{n=0}^{\infty} A_n^{\delta,\beta} z^n \tag{1.1}$$

which is valid for small z, $(0<z<1)$; $\delta > -1$ and $a \geqslant 2$. It is known [6] that for large n,

$$A_n^{\delta,\beta} \sim \frac{n^{\delta}}{\Gamma(\delta+1)}(\log n)^{\beta}, \quad \text{for} \quad \delta \neq -1, -2, -3\ldots \tag{1.2}$$

$$A_n^{\delta,\beta} \sim (-1)^{\delta-1}(|\delta|-1)!\,\beta n^{\delta}(\log n)^{\beta-1}, \quad \text{for } \delta = -1, -2, -3\ldots \tag{1.3}$$

Let ΣA_n be an infinite series with sequence of its partial sums $\{s_n\}$. Then sequence to sequence transformation

$$\sigma_n^{\delta,\beta} = \frac{1}{A_n^{\delta,\beta}} \sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1,\beta} s_{\nu}$$

is called the (C,δ,β) mean of the series ΣA_n.

If $\delta_n^{\delta,\beta} \to s$ as $n \to \infty$, then we say that the sequence $\{s_n\}$ or the series ΣA_n is summable (C,δ,β) to s. If $\{\sigma_n^{\delta,\beta}\}$ is a function of bounded variation, that is to say,

$$\sum_{n=1}^{\infty} |\sigma_n^{\delta,\beta} - \sigma_n^{\delta-1,\beta}| < \infty,$$

we say that $\{s_n\}$ or the series ΣA_n is absolutely generalised Cesàro summable or summable $|C,\delta,\beta|$. It is easily seen that the generalized absolute Cesàro summability is a regular method for all finite β. Also, if $\beta = 0$, then $|C,\delta,\beta|$ summability is same as absolute Cesàro summability (C,δ) [2] and if $\beta<0$, then $|C,\delta,\beta| \Rightarrow |C,\delta|$. Thus $|C,\delta,\beta|$

may be regarded as the generalisation of the absolute Cesàro summability. Further, $|C,0,1|$ is equivalent to harmonic summability.

2. Notations

Let $f(t)$ be a periodic function with period 2π and integrable-L over $(-\pi,\pi)$ and let

$$f(t)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}A_n(t). \tag{2.1}$$

Then the series

$$\sum_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx)=\sum_{n=1}^{\infty}B_n(x); \tag{2.2}$$

is said to be the conjugate series of the Fourier series (2·1) at $t=x$. In what follows we use the following notations:

$$\psi(t)=\frac{1}{2}\{f(x+t)-f(x-t)\};$$

$$\bar{S}_k^{1+\delta,\beta}(n,t)=\sum_{\nu=0}^{k}A_{n-\nu}^{\delta,\beta}(\nu+1)\sin(\nu+1)t,\quad\text{for}\quad 0\leqslant k\leqslant n;$$

$$\bar{g}^{1+\delta,\beta}(n,t)=\frac{1}{A_n^{1+\delta,\beta}}\bar{S}_n^{1+\delta,\beta}(n,t);$$

$$\bar{E}^{1+\delta,\beta}(n,t)=\frac{1}{A_n^{1+\delta,\beta}}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta,\beta}\frac{(\nu+1)\sin(\nu+1)t}{\log(\nu+2)};$$

$$h(t)=\frac{\psi(t)}{\log(a/t)},\quad\text{for}\quad a>\pi;$$

$$\Delta u_n=u_n-u_{n+1},\ \text{for any sequence } \{u_n\};$$

$$\tau=[a/t];$$

and $[x]$ denotes the greatest integer not exceeding x.

3. Theorem

In 1937, Bosanquet and Hyslop [1] proved the following theorem.

Theorem A. *If* $|\psi(t)|/t$ *is integrable in* $(0,\pi)$ *then the conjugate series of* $f(t)$, *at* $t=x$ *is summable* $|C,1+\delta|$, $\delta>0$.

Subsequently, Mohanty and Ray [5] proved a theorem on the behaviour of a series associated with the conjugate series of a Fourier series. In this paper, they have shown that the series $\Sigma_{n=1}^{\infty}[B_n(x)/\log(n+1)]$ behaves more or less like the conjugate series (2·2), as far as the theorem A is concerned, and the function $h(t)$ plays the role of $\psi(t)$. The proved the following theorem:

Theorem B. *If* $|h(t)|/t$ *is integrable in* $(0, \pi)$, *then the series* $\Sigma_{n=1}^{\infty}[B_n(x)/\log(n+1)]$ *is summable* $|C, 1+\delta|$, *for* $\delta > 0$.

After studying deeply the character of the above theorem, we proved the following theorem:

Theorem. *If* $|h(t)|/t$ *is integrable in* $(0, \pi)$ *then the series* $\Sigma_{n=1}^{\infty}[B_n(x)/\log(n+1)]$ *is summable* $|C, 1+\delta, \beta|$ *for all real* β *and* $\delta > 0$.

4. Lemmas

We require the following lemmas for the proof of our theorem.

Lemma 1. (Das and Mohapatra [3]). *If* $T_n^{\delta,\beta}$ *denotes the* (C, δ, β) *mean of the sequence* $\{nA_n\}$ *then* $|C, \delta, \beta|$ *summability of the series* ΣA_n *is equivalent to*

$$\sum_{n=1}^{\infty} \frac{|T_n^{\delta,\beta}|}{n} < \infty.$$

Lemma 2. (Jena and Swamy [4]). *If*

$$\bar{g}^{1+\delta,\beta}(n,t) = \frac{1}{A_n^{1+\delta,\beta}} \bar{S}_n^{1+\delta,\beta}(n,t)$$

then

$$|\bar{g}^{1+\delta,\beta}(n,t)| = \begin{cases} 0(n), & \textit{for all } t, \\ 0\left(\dfrac{\tau^{1+\delta}(\log\tau)^{\beta}}{n^{\delta}(\log n)^{\beta}}\right), & \textit{for } t > (1/n). \end{cases}$$

Lemma 3.
If

$$\bar{S}_n^{1+\delta,\beta}(n,t) = \sum_{\nu=0}^{n} \mathrm{A}_{n-\nu}^{\delta,\beta}(\nu+1)\sin(\nu+1)t$$

then

$$|\bar{S}_n^{1+\delta,\beta}(n,t)| = \begin{cases} 0(n^{2+\delta}(\log n)^{\beta}), & \textit{for all } t, \\ 0(n\tau^{1+\delta}(\log\tau)^{\beta}), & \textit{for all } t > (1/n). \end{cases}$$

Proof.
Since

$$|\bar{S}_n^{1+\delta,\beta}(n,t)| = \left|\sum_{\nu=0}^{n} A_{n-\nu}^{\delta,\beta}(\nu+1)\sin(\nu+1)t\right|$$

we have

$$\begin{aligned} |\bar{S}_n^{1+\delta,\beta}(n,t)| &= |A_n^{1+\delta,\beta}|\left|\frac{1}{A_n^{1+\delta,\beta}}\sum_{\nu=0}^{n} A_{n-\nu}^{\delta,\beta}(\nu+1)\sin(\nu+1)t\right| \\ &= |A_n^{1+\delta,\beta}||\bar{g}^{1+\delta,\beta}(n,t)| \\ &= \begin{cases} 0(n^{2+\delta}(\log n)^{\beta}), & \textit{for all } t, \\ 0(n\tau^{1+\delta}(\log\tau)^{\beta}), & \textit{for } t > (1/n), \text{ by lemma 2.} \end{cases} \end{aligned}$$

Lemma 4.
If

$$|\bar{E}^{1+\delta,\beta}(n,t)| = \frac{1}{A_n^{1+\delta,\beta}} \sum_{\nu=0}^{n} A_{n-\nu}^{\delta,\beta} \frac{(\nu+1)\sin(\nu+1)t}{\log(\nu+2)}$$

then

$$|\bar{E}^{1+\delta,\beta}(n,t)| = \begin{cases} 0\left(\dfrac{n}{\log n}\right), & \textit{for all } t, \\ 0\left(\dfrac{\tau^{1+\delta}(\log\tau)^{\beta})}{n^{0}(\log n)^{1+\beta}}\right), & \textit{for } t > (1/n). \end{cases}$$

Proof.
For all t,

$$\begin{aligned} |\bar{E}^{1+\delta,\beta}(n,t)| &= \left| \frac{1}{A_n^{1+\delta,\beta}} \sum_{\nu=0}^{n} A_{n-\nu}^{\delta,\beta} \frac{(\nu+1)\sin(\nu+1)t}{\log(\nu+2)} \right| \\ &\leqslant \frac{1}{|A_n^{1+\delta,\beta}|} \sum_{\nu=0}^{n} \left| A_{n-\nu}^{\delta,\beta} \frac{(\nu+1)}{\log(\nu+2)} \right| \\ &= 0\left(\frac{n}{\log n}\right). \end{aligned}$$

For $t > (1/n)$,

$$\begin{aligned} |\bar{E}^{1+\delta,\beta}(n,t)| &= \left| \frac{1}{A_n^{1+\delta,\beta}} \sum_{\nu=0}^{n} A_{n-\nu}^{\delta,\beta} \frac{(\nu+1)\sin(\nu+1)t}{\log(\nu+2)} \right| \\ &= \left| \frac{1}{A_n^{1+\delta,\beta}} \left[\sum_{\nu=0}^{n-1} \bar{S}_\nu^{1+\delta,\beta}(n,t)\Delta\left\{\frac{1}{\log(\nu+2)}\right\} + \frac{\bar{S}_n^{1+\delta,\beta}(n,t)}{\log(\nu+2)} \right] \right. \end{aligned}$$

(by Abel's method of partial summation)

$$\begin{aligned} &= \frac{1}{A_n^{1+\delta,\beta}} 0\left[\sum_{\nu=0}^{n-1} \nu\tau^{1+\delta}(\log\tau)^{\beta}\Delta\left\{\frac{1}{\log(\nu+2}\right\} \right] \\ &\quad + \frac{1}{A_n^{1+\delta,\beta}} 0\left[\frac{n\tau^{1+\delta}(\log\tau)^{\beta}}{\log n} \right], \quad \text{by lemma 3} \\ &= 0\left[\frac{\tau^{1+\delta}(\log\tau)^{\beta}}{n^{1+\delta}(\log n)^{\beta}} \left(\sum_{\nu=0}^{n-1} \nu\Delta\left\{\frac{1}{\log(\nu+2)}\right\} \right) \right] \\ &\quad + 0\left[\frac{n\tau^{1+\delta}(\log\tau)^{\beta}}{n^{1+\delta}(\log n)^{\beta}(\log n)} \right] \\ &= 0\left[\frac{\tau^{1+\delta}(\log\tau)^{\beta}}{n^{\delta}(\log n)^{1+\beta}} \right]. \end{aligned}$$

5. Proof of the theorem

Let

$$T_n^{1+\delta,\beta} = \frac{1}{A_n^{1+\delta,\beta}} \sum_{\nu=0}^{n} A_{n-\nu}^{\delta,\beta} \frac{(\nu+1)B_{\nu+1}(t)}{\log(\nu+2)}.$$

Then $T_n^{1+\delta,\beta}$ is the nth Generalised Cesàro mean of order $(1+\delta,\beta)$ of the sequence $\{(\nu+1)B_{\nu+1}(t)/\log(\nu+2)\}$. After lemma 1, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{|T_n^{1+\delta,\beta}|}{n}<\infty.$$

Since

$$B_{\nu+1}(t)=\frac{2}{\pi}\int_0^{\pi}\psi(t)\sin(\nu+1)t\,dt,$$

we have

$$\begin{aligned}T_n^{1+\delta,\beta}&=\frac{1}{A_n^{1+\delta,\beta}}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta,\beta}\frac{(\nu+1)}{\log(\nu+1)}\frac{2}{\pi}\int_0^{\pi}\psi(t)\sin(\nu+1)t\,\mathrm{d}t\\&=\frac{2}{\pi}\int_0^{\pi}\psi(t)\frac{1}{A_n^{1+\delta,\beta}}\sum_{\nu=0}^{n}A_{n-\nu}^{\delta,\beta}\frac{(\nu+1)\sin(\nu+1)t}{\log(\nu+2)}\mathrm{d}t\\&=\frac{2}{\pi}\int_0^{\pi}\psi(t)\bar{E}^{1+\delta,\beta}(n,t)\mathrm{d}t\\&=\frac{2}{\pi}\int_0^{\pi}\frac{\psi(t)}{t\log(a/t)}\bar{E}^{1+\delta,\beta}(n,t)t\log(a/t)\mathrm{d}t\\&=\frac{2}{\pi}\int_0^{\pi}\frac{h(t)}{t}\bar{E}^{1+\delta,\beta}(n,t)t\log(a/t)\mathrm{d}t.\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{|T_n^{1+\delta},\beta|}{n}\\&=\sum_{n=1}^{\infty}\frac{\left|\frac{2}{\pi}\int_0^{\pi}\frac{h(t)}{t}\bar{E}^{1+\delta,\beta}(n,t)t\log(a/t)\mathrm{d}t\right|}{n}\\&\leqslant\frac{2}{\pi}\int_0^{\pi}\frac{|h(t)|}{t}\mathrm{d}t\sum_{n=1}^{\infty}\frac{|\bar{E}^{1+\delta,\beta}(n,t)|}{n}(t\log(a/t)).\end{aligned}$$

Since according to the hypothesis $|h(t)|/t$ is integrable in $(0,\pi)$,

$$\int_0^{\pi}\frac{|h(t)|}{t}\mathrm{d}t<\infty$$

Hence our theorem is proved if we can show that

$$\sum_{n=1}^{\infty}\frac{|\bar{E}^{1+\delta,\beta}(n,t)|}{n}=0[t^{-1}(\log(a/t))^{-1}],\text{ uniformly in }0<t\leqslant\pi.$$

Now

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{|\bar{E}^{1+\delta,\beta}(n,t)|}{n}\\&=\sum_{n=1}^{\tau}\frac{|\bar{E}^{1+\delta,\beta}(n,t)|}{n}+\sum_{n=\tau+1}^{\infty}\frac{|\bar{E}^{1+\delta,\beta}(n,t)|}{n},\\&=\Sigma_1+\Sigma_2,\text{ say.}\end{aligned}$$

Thus

$$\begin{aligned}
\Sigma_1 &= \sum_{n=1}^{\tau} n^{-1}|\bar{E}^{1+\delta,\beta}(n,t)| \\
&= \sum_{n=1}^{\tau} \frac{1}{n} 0\left(\frac{n}{\log n}\right), \text{ by lemma 4} \\
&= \sum_{n=1}^{\tau} 0\left(\frac{1}{\log n}\right) \\
&= 0(t^{-1}(\log\tau)^{-1}).
\end{aligned}$$

Again

$$\begin{aligned}
\Sigma_2 &= \sum_{n=\tau+1}^{\infty} n^{-1}|\bar{E}^{1+\delta,\beta}(n,t)| \\
&= \sum_{n=\tau+1}^{\infty} n^{-1}\left[0\left(\frac{\tau^{1+\delta}(\log\tau)^{\beta}}{n^{\delta}(\log n)^{1+\beta}}\right)\right] \text{ by lemma 4} \\
&= 0(\tau^{1+\delta}(\log\tau)^{\beta}) \sum_{n=\tau+1}^{\infty} \frac{1}{n^{1+\delta}(\log n)^{1+\beta}} \\
&\leqslant 0(\tau^{1+\delta}(\log\tau)^{\beta}) \frac{1}{(\log\tau)^{1+\beta}} \sum_{n=\tau+1}^{\infty} \frac{1}{n^{1+\delta}}, \quad \text{for } \beta \geqslant 0 \\
&= 0(\tau^{1+\delta}(\log\tau)^{-1})(\tau^{-\delta}), \quad \text{for } \delta > 0 \\
&= 0(t^{-1}(\log\tau)^{-1})
\end{aligned}$$

and for $\beta < 0$,

$$\begin{aligned}
\Sigma_2 &= 0\left(\frac{\tau^{1+\delta}}{\log\tau)^{\beta}}\right) \sum_{n=\tau+1}^{\infty} \frac{(\log n)^{\beta-1}}{n^{1+\delta}} \\
&= 0\left(\frac{\tau^{1+\delta}}{(\log\tau)^{\beta}}\right) \sum_{n=\tau+1}^{\infty} \frac{(\log n)^{\beta-1}}{n^{\delta/2}} \cdot \frac{1}{n^{1+\delta/2}} \\
&\leqslant 0\left(\frac{\tau^{1+\delta}}{(\log\tau)^{\beta}} \frac{(\log\tau)^{\beta-1}}{\delta/2}\right) \sum_{n=+1}^{\infty} \frac{1}{n^{1+\delta/2}} \\
&= 0(t^{-1}(\log\tau)^{-1}), \quad \text{for } \delta > 0.
\end{aligned}$$

This completes the proof of the theorem.

6. Remarks

Setting $\beta = 0$, the theorem of Mohanty and Ray comes out as a particular case of our theorem. Also, since for $\beta < 0$, $|C, 1+\delta, \beta| \Rightarrow |C, 1+\delta|$, our theorem goes still deeper and derives the theorem of Mohanty and Ray as a corollary.

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On regularity of twisted spherical means and special Hermite expansions

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Abstract. Regularity properties of twisted spherical means are studied in terms of certain Sobolev spaces defined using Laguerre functions. As an application we prove a localisation theorem for special Hermite expansions.

Keywords. Spherical means; twisted convolution; Sobolev spaces; special Hermite series.

1. Introduction

Regularity of the spherical means $f(x,t)$ defined on $\mathbb{R}^n$ for f in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by the equation

$$f(x,t)=\int_{S^{n-1}} f(x-ty)\mathrm{d}\sigma(y) \tag{1.1}$$

has been the subject of several papers. In 1976, Stein [10] proved that when $n\geqslant 3$ and $p>n/(n-1)$ the spherical means $f(x,t)$ of a function f in $L^p(\mathbb{R}^n)$ are bounded and continuous as a function of t for almost every x in $\mathbb{R}^n$. Later more refined results were obtained by Oberlin–Stein [4], Peyriere–Sjolin [7] and Sjolin [8]. To wit, the following theorem has been proved by Sjolin [8]. Let $H^s(\mathbb{R})$ stand for the usual L^2 Sobolev space of order s and let $\varphi\in C_0^\infty(0,\infty)$. Then the following is true.

Theorem (Sjolin). *If* $n\geqslant 2$, $2n/2n-1\leqslant p\leqslant 2$ *and* $s=n(1-1/p)-(1/2)$ *then for any* f *in* $L^p(\mathbb{R}^n)$ *one has*

$$\left(\int_{\mathbb{R}^n}\|\varphi(.)f\text{[illegible]})\|^2_{H^s}\mathrm{d}x\right)^{1/2}\leqslant C\left(\int|f(x)|^p\mathrm{d}x\right)^{1/p}.$$

The above result has been extended from $\mathbb{R}^n$ to any compact symmetric space by Colzani [1]. His study of the spherical means on these spaces is essentially group theoretic and is based on the theory of spherical functions. In this setting the proofs turned out to be simpler and the results quite general. In order to state the main result of Colzani it is necessary to recall the definition of the spherical means on a compact symmetric space.

Let (G,K) be a compact symmetric pair and $X=G/K$ be the associated symmetric space. Functions on X are identified with right K-invariant functions on G. Let $G=KAK$ be a Cartan decomposition of G and let $\mathrm{d}g=\mathrm{d}k\ w(t)\mathrm{d}t\,\mathrm{d}k'$ be the corresponding decomposition of the Haar measure $\mathrm{d}g$ on G. Here $w(t)$ is an appropriate weight function defined on A. If $f \in L^1(G)$ we define the spherical means $f(g,t)$, $g \in G$ and $t \in A$ by

$$f(g,t) = \int_K \int_K f(gktk')\,\mathrm{d}k\,\mathrm{d}k'. \tag{1.2}$$

Clearly, $f(g,t)$ is K-invariant as a function of g.

Using the Peter-Weyl theorem we can expand $f(g,t)$ in terms of zonal spherical functions. To see this let us recall the Peter-Weyl theorem for right K-invariant functions of G. Let $\hat{G}_K$ be the subset of the dual of G consisting of equivalence classes of class 1 representations of G and let $\{\varphi_\lambda\}$, $\lambda \in \hat{G}_K$ be the associated system of zonal spherical functions. Then the Peter-Weyl theorem reads

$$f = \sum_{\lambda \in \hat{G}_K} d_\lambda f * \varphi_\lambda, \tag{1.3}$$

where d_λ stands for the dimension of λ. Since the spherical functions φ_λ satisfy the relation

$$\int_K \varphi_\lambda(gkg')\,\mathrm{d}k = \varphi_\lambda(g)\varphi_\lambda(g') \tag{1.4}$$

the spherical means $f(g,t)$ have the expansion

$$f(g,t) = \sum_{\lambda \in \hat{G}_K} d_\lambda f * \varphi_\lambda(g)\varphi_\lambda(t). \tag{1.5}$$

The above expression for $f(g,t)$ suggests that it is natural to study the regularity properties of $f(g,t)$ in terms of some sort of Sobolev spaces associated to the system $\{\varphi_\lambda\}$. For $t \in A$ we get $\psi_\lambda(t) = d_\lambda^{1/2}\varphi_\lambda(t)$ so that $\{\psi_\lambda\}$ is an orthonormal basis for $L^2(A, w(t)\mathrm{d}t)$. We then introduce the Sobolev space $L_s^2(A, w(t)\mathrm{d}t)$ to be the set of all functions $f = \Sigma \hat{f}(\lambda)\psi_\lambda$ for which the norm

$$\|f\|_{L_s^2}^2 = \sum_{\lambda \in \hat{G}_K} (1 + |\lambda|^2)^s |\hat{f}(\lambda)|^2 \tag{1.6}$$

is finite. Here $|\lambda|$ denotes the norm of the highest weight corresponding to λ.

Measuring the regularity of the spherical means $f(g,t)$ in terms of the above Sobolev spaces Colzani established the following.

Theorem (Colzani). *Let $X = G/K$ be a compact rank one symmetric space of real dimension n. If $f \in L^p(X)$, $1 < p \leqslant 2$, and $s = n(1-(1/p)) - (1/2)$ then*

$$\int_G \|f(g,.)\|_{L_s^2}^2 \mathrm{d}g \leqslant C \|f\|_p^2.$$

Using the above regularity theorem he was able to establish a localisation theorem for spherical harmonic expansions of L^2 functions on compact rank one symmetric spaces.

Our goal in this paper is to establish similar regularity results for the twisted spherical means on $\mathbb{C}^n$ and to prove an almost everywhere convergence result for the twisted spherical means. Before stating the results we need to set the notations up. The twisted convolution $f \times g$ of two functions f and g defined on $\mathbb{C}^n$ is the function

$$f \times g(z) = \int_{\mathbb{C}^n} f(z-w)g(w)\exp\left(\frac{i}{2}\operatorname{Im} z.\bar{w}\right)\mathrm{d}w. \tag{1.7}$$

If μ_r stands for the normalized surface measure on the sphere $\{z \in \mathbb{C}^n : |z| = r\}$ the twisted spherical means is defined to be

$$f \times \mu_r(z) = \int_{|w|=r} f(z-w)\exp\left(\frac{i}{2}\operatorname{Im} z.\bar{w}\right)\mathrm{d}\mu_r. \tag{1.8}$$

In this paper we are interested in the regularity of these twisted spherical means.

The study of the twisted spherical means turns out to be much simpler than the study of spherical means; thanks to the existence of an analogue of the Peter-Weyl theorem. This analogue is the so-called special Hermite expansion which is written as

$$f = (2\pi)^{-n}\sum_{k=0}^{\infty} f \times \varphi_k. \tag{1.9}$$

Here φ_k stands for the Laguerre function

$$\varphi_k(z) = L_k^{n-1}(\tfrac{1}{2}|z|^2)\exp(-(1/4)|z|^2) \tag{1.10}$$

where L_k^{n-1} is the kth Laguerre polynomial of type $(n-1)$. For various results concerning the special Hermite expansions we refer to Strichartz [11] and Thangavelu [14] and also to the monographs by Folland [2] and Thangavelu [16].

We also have an analogue of the expansion (1.5) for the twisted spherical means, namely,

$$f \times \mu_r(z) = (2\pi)^{-n}\sum_{k=0}^{\infty}\frac{k!(n-1)!}{(k+n-1)!}\varphi_k(r) f \times \varphi_k(z). \tag{1.11}$$

This has been proved in [14] and has played a crucial role in the study of spherical means on the Heisenberg group (see [14]). The above expansion suggests that we should measure the regularity of $f \times \mu_r$ in terms of some sort of Sobolev spaces defined using the functions $\varphi_k(r)$. If we define

$$\psi_k(r) = \left(\frac{2^{1-n}k!}{(k+n-1)!}\right)^{1/2}\varphi_k(r) \tag{1.12}$$

then the system $\{\psi_k\}$ forms an orthonormal basis for $L^2(\mathbb{R}_+, r^{2n-1}\mathrm{d}r)$ where $\mathbb{R}_+ = (0,\infty)$. We introduce the Sobolev spaces $W_R^s(\mathbb{R}_+)$ to be the set of all functions of the form $f = \Sigma_{k=0}^{\infty} a_k\psi_k$ for which the norms

$$|f|_s^2 = \sum_{k=0}^{\infty}|a_k|^2(2k+n)^{2s} \tag{1.13}$$

are finite. For the twisted spherical means we then establish the following theorem.

Theorem A. *If $f \in L^p(\mathbb{C}^n)$, $1 < p \leqslant 2$, $s = n(1-(1/p)) - (1/2)$ then one has the inequality and*

$$\int_{\mathbb{C}^n} |f \times \mu_r(z)|_s^2 \, dz \leqslant C \|f\|_p^2.$$

The Sobolev space $W_R^s(\mathbb{R}_+)$ turns out to be a subspace of the Laguerre (or twisted) Sobolev space $W_L^s(\mathbb{C}^n)$. This space and another Sobolev space $W_H^s(\mathbb{R}^n)$ (Hermite Sobolev space) will be introduced in the next section. It turns out that $f \in W_R^s(\mathbb{R}_+)$ if and only if the function g defined by $g(z) = f(|z|)$ belongs to $W_L^s(\mathbb{C}^n)$. We also prove the embedding theorem for the Sobolev space W_L^s and W_H^s. As a consequence we also establish a pointwise convergence theorem for the twisted spherical means.

We also study the localisation problem for the special Hermite expansion (1.9). Let $S_N f(z)$ be the partial sums defined by

$$S_N f(z) = (2\pi)^{-n} \sum_{k=0}^{N} f \times \varphi_k(z). \tag{1.14}$$

Suppose we know that $f(w)$ vanishes in a neighborhood of z, say in $|z - w| \leqslant a$. We are interested in finding conditions of f which will ensure that $S_N f(z) \to 0$ as $N \to \infty$ for a.e. z in $|z - w| \leqslant a$. Using the regularity of spherical means for L^2 functions we are able to prove the following theorem.

Theorem B. *Assume $n \geqslant 2$ and f is a compactly supported function in $W_L^{1/2}(\mathbb{C}^n)$. If $f(z) = 0$ on an open set Ω, then $S_N f(z) \to 0$ for a.e. z in Ω as $N \to \infty$.*

This theorem is proved in § 4. We introduce the Sobolev spaces in the next section. Regularity theorems for the spherical means are proved in § 3.

2. Invariant Sobolev spaces

We have introduced the Laguerre Sobolev spaces $W_L^s(\mathbb{C}^n)$ in [14] in connection with the spherical means on the Heisenberg group. There we have called them twisted Sobolev spaces. They have been also introduced in Peetre-Sparr [6] in connection with noncommutative interpolation. Let L be the special Hermite operator defined by

$$L = -\Delta + \frac{1}{4}|z|^2 - i \sum_{j=1}^{n} \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right). \tag{2.1}$$

Then the special Hermite functions are eigenfunctions of the operator L and the series (1.9) is nothing but the eigenfunction expansion associated to L. In fact one has

$$L(f \times \varphi_k) = (2k+n) f \times \varphi_k \tag{2.2}$$

for any k. In view of this and spectral theorem one can define L^s by

$$L^s f = (2\pi)^{-n} \sum_{k=0}^{\infty} (2k+n)^s f \times \varphi_k. \tag{2.3}$$

The usual Sobolev space $H^s(\mathbb{R}^n)$ is defined to be the image of $L^2(\mathbb{R}^n)$ under the operator $(-\Delta + 1)^{-(s/2)}$. Similarly, we define $W_L^s(\mathbb{C}^n)$ to be the image of $L^2(\mathbb{C}^n)$ under

L^{-s}. In other words, $f \in W_L^s$ if and only if

$$|f|_s^2 = (2\pi)^{-2n} \sum_{k=0}^{\infty} (2k+n)^{2s} \|f \times \varphi_k\|_2^2 \tag{2.4}$$

is finite. Before studying properties of W_L^s we also introduce the Sobolev spaces W_H^s.

Consider the Hermite operator $H = -\Delta + (1/4)|x|^2$ on $\mathbb{R}^n$ for which the eigenfunctions are the ordinary Hermite functions $\Phi_\mu(x)$. If $P_k f$ stand for the projection of f in $L^2(\mathbb{R}^n)$ into the kth eigenspace spanned by $\{\Phi_\mu : |\mu| = k\}$ then the spectral decomposition of H is

$$Hf = \sum_{k=0}^{\infty} (2k+n) P_k f. \tag{2.5}$$

Using this one can define the operators H^s and $W_H^s(\mathbb{R}^n)$ is defined to be the image of $L^2(\mathbb{R}^n)$ under H^{-s}. This means f belongs to $W_H^s(\mathbb{R}^n)$ if and only if

$$|f|_s^2 = \sum_{k=0}^{\infty} (2k+n)^{2s} \|P_k f\|_2^2 \tag{2.6}$$

is finite. We use the same notation for the norms in W_L^s as well as in W_H^s. The situation will make it clear which space we refer to.

The Hermite and Laguerre Sobolev spaces enjoy an invariant property which is not shared by the usual Sobolev spaces $H^s(\mathbb{R}^n)$. Namely, W_H^s is invariant under the Fourier transform and W_L^s invariant under the symplectic Fourier transform. The invariance of W_H^s under the Fourier transform follows from the fact that $F(P_k f) = (-i)^k P_k f$ where Ff is the Fourier transform of f. This follows from the well known properties of the Hermite functions. The invariance of W_L^s can be seen as follows.

Let $F_s f$ be the symplectic Fourier transform of f which is defined by

$$F_s f(z) = \int f(w) \exp\left(-\frac{i}{2} \operatorname{Im} z \cdot \bar{w}\right) dw. \tag{2.7}$$

We can write this as $F_s f(z) = f \times 1(z)$. In view of this

$$F_s f \times \varphi_k = f \times (1 \times \varphi_k) \tag{2.8}$$

and if we can show that

$$1 \times \varphi_k = (-1)^k \varphi_k \tag{2.9}$$

the invariance of W_L^s under F_s will follow immediately. But (2.9) is an easy consequence of the generating function identity

$$\sum_{k=0}^{\infty} r^k \varphi_k(z) = (1-r)^{-n} \exp(-(1/4)(1+r/1-r)|z|^2) \tag{2.10}$$

valid for any $0 \leqslant r < 1$.

We now bring out the relation between the Hermite Sobolev space W_H^s and the ordinary Sobolev space H^s. If we let

$$A_j = \left(\frac{\partial}{\partial x_j} + \frac{1}{2} x_j\right), \quad j = 1, 2, \ldots n. \tag{2.11}$$

and A_j^* the adjoint of A_j then it is easily seen that

$$H=\left(-\Delta+\frac{1}{4}|x|^2\right)=\frac{1}{2}\sum_{j=1}^{n}(A_jA_j^*+A_j^*A_j).$$

In view of this it is clear that $H^m f$ is a finite linear combination of terms of the for $x^\alpha\partial_x^\beta f$ where $|\alpha|+|\beta|\leqslant 2m$ when m is an integer. Thus, in this case $W_H^m(\mathbb{R}^n)\subset H^{2m}(\mathbb{R}$ This inclusion is true for any $m\geqslant 0$.

Theorem 2.1 (i) *If* $s\geqslant 0$ *then* $W_H^s(\mathbb{R}^n)\subset H^{2s}(\mathbb{R}^n)$. (ii) *Conversely, if* $s\geqslant 0$ *and* $\varphi\in C_0^\infty(\mathbb{R}$ *then there is a constant* $C=C(\varphi)$ *depending on* φ *such that*

$$\int|H^s(\varphi f)|^2\mathrm{d}x\leqslant C\int|(-\Delta+1)^s f|^2\mathrm{d}x.$$

Proof. In order to prove (i) it is enough to show that the operator $(-\Delta+1)^s(-\Delta+|x|^2)$ is bounded on $L^2(\mathbb{R}^n)$. In [15] we have proved that $(-\Delta+|x|^2)^{-s}$ is a pseuc differential operator whose symbol $\sigma(x,\xi)$ satisfies the estimates

$$|\partial_x^\alpha\partial_\xi^\beta\sigma(x,\xi)|\leqslant C(1+|x|+|\xi|)^{-2s-|\alpha|-|\beta|}, \qquad (2.1$$

where C is independent of x. Therefore, it follows that $(-\Delta+1)^s(-\Delta+|x|)^2)^{-s}$ is pseudo differential operator with symbol in $S_{1,0}^0$ and hence is bounded on $L^2(\mathbb{R}^n)$

To establish (ii) we look at the operator $(-\Delta+|x|^2)^s\varphi(x)(-\Delta+1)^{-s}$. Aga because of the compactly supported φ, the above operator is a pseudo different operator whose symbol b satisfies

$$|\partial_x^\alpha\partial_\xi^\beta b(x,\xi)|\leqslant C_{\alpha\beta}(\varphi)(1+|\xi|)^{-|\beta|}$$

where $C_{\alpha\beta}(\varphi)$ is independent of x. This proves part (ii).

We remark that a similar inclusion is true for the Laguerre Sobolev spaces als This follows from the relation

$$L=-\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j+\bar{Z}_jZ_j) \qquad (2.1$$

where the vector fields Z_j and $\bar{Z}_j$ are defined by

$$Z_j=\left(\frac{\partial}{\partial z_j}+\frac{1}{2}\bar{z}_j\right),\bar{Z}_j=\left(\frac{\partial}{\partial\bar{z}_j}-\frac{1}{2}z_j\right). \qquad (2.$$

The proof runs along similar lines and we leave the details to the interested read

We now turn our attention to the relation between the Sobolev spaces W_L^s, I and W_R^s introduced in § 1. If for any space V of functions on $\mathbb{C}^n\simeq\mathbb{R}^{2n}$, $(V)_R$ stan for the subspace of all radial functions in V then we have the following relation:

$$(W_L^s(\mathbb{C}^n))_R=(W_H^s(\mathbb{R}^{2n}))_R=W_R^s(\mathbb{R}_+). \qquad (2.$$

To see the first equality we observe that when f is radial $Lf=Hf$. This follows fr the fact that f is annihilated by operator

$$\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j}-y_j\frac{\partial}{\partial x_j}\right).$$

This observation has been used in the study of Hermite expansions on $\mathbb{R}^{2n}$ for radial functions. Thus as far as radial functions are concerned L and H have the same spectral decomposition and hence the equality $(W_L^s)_R=(W_H^s)_R$.

The equality $W_R^s(\mathbb{R}_+)=(W_L^s(\mathbb{C}^n))_R$ follows from the following fact. When f is a radial function of $\mathbb{C}^n$ then the special Hermite expansion of f reduces to the Laguerre expansion. More precisely, one has

$$f\times\varphi_k(z)=c_n(f,\psi_k)\psi_k(r). \tag{2.16}$$

It is now obvious from this relation that $W_R^s(\mathbb{R}_+)$ is equal to $(W_L^s)_R$. With these observations we end discussion on the various Sobolev spaces and turn our attention to embedding theorems.

Theorem 2.2 (i) *If* $s>\frac{n}{4}$, $W_H^s(\mathbb{R}^n)\subset L^\infty(\mathbb{R}^n)\cap C(\mathbb{R}^n)$. (ii) *If* $s>\frac{n}{2}$, $W_L^s(\mathbb{C}^n)\subset L^\infty(\mathbb{C}^n)\cap C(\mathbb{C}^n)$. *In both cases the inclusion is continuous.*

Proof. We first Prove (i). Writing

$$f(x)=\sum_\mu \hat{f}(\mu)\Phi_\mu(x), \tag{2.17}$$

where $\hat{f}(\mu)=(f,\Phi_\mu)$ and applying Cauchy-Schwarz we have

$$|f(x)|^2\leqslant\left(\sum_\mu(2|\mu|+n)^{2s}|\hat{f}(\mu)|^2\right)B(x) \tag{2.18}$$

where

$$B(x)=\sum_\mu(2|\mu|+n)^{-2s}|\Phi_\mu(x)|^2. \tag{2.19}$$

The sum on the right hand side of (2.18) is nothing but $|f|_s^2$ and consequently

$$\sup_x|f(x)|\leqslant C|f|_s \tag{2.20}$$

would follow once we show that $B(x)$ is bounded.

That $B(x)$ is bounded can be proved in two ways. Writing $\Phi_k(x,x)=\sum_{|\mu|=k}(\Phi_\mu(x))^2$ we see that

$$B(x)=\sum_{k=0}^{\infty}(2k+n)^{-2s}\Phi_k(x,x). \tag{2.21}$$

In [13] we have established the estimate

$$\Phi_k(x,x)\leqslant C(2k+n)^{(n/2)-1} \tag{2.22}$$

with C independent of x. Therefore,

$$B(x) \leqslant C \sum_{k=0}^{\infty} (2k+n) - 2s + (n/2) - 1 \tag{2.23}$$

the sum being convergent as $s > n/4$. We can also prove the boundedness of $B(x)$ without using the estimate (2.22) but using the generating function for the Hermite functions. One has

$$\sum_{k=0}^{\infty} r^k \Phi_k(x,x) = \pi^{-(n/2)}(1-r)^{-(n/2)} \exp\left(-\left(\frac{1}{2}\right)\left(\frac{1-r}{1+r}|x|^2\right)\right). \tag{2.24}$$

Taking $r = e^{-2t}$ we obtain

$$\sum_{k=0}^{\infty} \exp(-(2k+n)t)\Phi_k(x,x) = (2\pi)^{-(n/2)}(\sinh 2t)^{-(n/2)}$$

$$\exp\left(-\left(\frac{1}{2}\right)(\tanh t)|x|^2\right) \tag{2.25}$$

Integrating the above identity against $t^{2s-1}\mathrm{d}t$ we get

$$\begin{aligned} &\sum_{k=0}^{\infty} (2k+n)^{-2s}\Phi_k(x,x) \\ &= \frac{(2\pi)^{-(n/2)}}{\Gamma(2s)} \int_0^{\infty} (\sinh 2t)^{-(n/2)} t^{2s-1} \exp\left(-\left(\frac{1}{2}\right)(\tanh t)|x|^2\right)\mathrm{d}t \\ &\leqslant \frac{(2\pi)^{-(n/2)}}{\Gamma(2s)} \int_0^{\infty} t^{2s-1} \exp(-(nt))(1-\exp(-4t))^{-(n/2)}\mathrm{d}t \leqslant C \end{aligned} \tag{2.26}$$

since $s > (n/4)$.

We prove (ii) using an interesting identity satisfied by the special Hermite functions. For each pair of multi-indices μ and ν the special Hermite function $\Phi_{\mu\nu}$ is defined by

$$\Phi_{\mu\nu}(z) = (2\pi)^{-(n/2)} \int \exp(ix\cdot\zeta)\Phi_{\mu}\left(\zeta + \frac{1}{2}y\right)\Phi_{\nu}\left(\zeta - \frac{1}{2}y\right)\mathrm{d}\zeta. \tag{2.27}$$

These functions have very interesting properties. To wit, one has

$$\Phi_{\mu\nu} \times \Phi_{\alpha\beta} = (2\pi)^{n/2}\delta_{\nu\alpha}\Phi_{\mu\beta}. \tag{2.28}$$

They form an orthonormal basis for $L^2(\mathbb{C}^n)$ and one has

$$f \times \varphi_k = (2\pi)^n \sum_{|\nu|=k} \sum_{\mu} (f, \Phi_{\mu\nu})\Phi_{\mu\nu}. \tag{2.29}$$

For all these and other interesting properties we refer to the monograph [16].

The identity which we need is the following.

PROPOSITION 2.1

For any z in $\mathbb{C}^n$ one has

$$\sum_{\beta} |\Phi_{\alpha\beta}(z)|^2 = (2\pi)^{-n}.$$

Proof. The functions $\{\bar{\Phi}_{\nu\beta}(w)\}$ form an orthonormal basis for $L^2(\mathbb{C}^n)$. We expand the function

$$\Phi_{\alpha\mu}(z-w)\exp(i/2)(\operatorname{Im} z.\bar{w}).$$

in terms of $\Phi_{\nu\beta}$. We have

$$\int \Phi_{\alpha\mu}(z-w)\exp(i/2)(\operatorname{Im} z\cdot\bar{w})\Phi_{\nu\beta}(w)dw = \Phi_{\alpha\mu}\times\Phi_{\nu\beta}(z) = \delta_{\mu\nu}(2\pi)^{n/2}\Phi_{\alpha\beta}(z).$$

Thus we have

$$\Phi_{\alpha\mu}(z-w)\exp(i/2)(\operatorname{Im} z.\bar{w}) = (2\pi)^{n/2}\sum_{\beta}\Phi_{\alpha\beta}(z)\bar{\Phi}_{\mu\beta}(w).$$

The orthonormality of the functions $\Phi_{\mu\beta}$ now proves the proposition.

We can now complete the proof of theorem 2.2. Writing

$$f(z) = \sum_{\mu}\sum_{\nu}(f,\Phi_{\mu\nu})\Phi_{\mu\nu}(z) \tag{2.30}$$

we obtain the inequality

$$|f(z)|^2 \leqslant AB(z) \tag{2.31}$$

where

$$\begin{aligned} A &= \sum\sum(2|\nu|+n)^{2s}|(f,\Phi_{\mu\nu})|^2, \\ B &= \sum\sum(2|\nu|+n)^{-2s}|\Phi_{\mu\nu}(z)|^2. \end{aligned} \tag{2.32}$$

In view of the relation (2.29) we get

$$A \leqslant (2\pi)^{-2n}|f|_s^2. \tag{2.33}$$

From the definition of $\Phi_{\mu\nu}$ it is clear that $\bar{\Phi}_{\mu\nu}(-z) = \Phi_{\nu\mu}(z)$ and hence

$$B(z) = \sum_{\nu}(2|\nu|+n)^{-2s}\sum_{\mu}|\Phi_{\mu\nu}(z)|^2 \leqslant (2\pi)^{-n}\sum_{k=0}^{\infty}(2k+n)^{-2s+n-1}$$

in view of the proposition. The last sum is finite precisely when $s > n/2$. Hence we have proved

$$|f(z)| \leqslant C|f|_s$$

for all z in $\mathbb{C}^n$. Hence the theorem.

3. Regularity of twisted spherical means

In this section we study the regularity properties of the spherical means $f\times\mu_r$ as a function of r in terms of the Sobolev spaces $W_R^s(\mathbb{R}_+)$. We first show that the spherical means are regularising in the following sense.

Theorem 3.1 *If $f \in W_L^s(\mathbb{C}^n)$ then for almost every $z \in \mathbb{C}^n$, $f \times \mu_r(z)$ belongs to $W_R^{s+(n-1)/2}(\mathbb{R}_+)$ and one has*

$$\int |f \times \mu_r(z)|^2_{s+(n-1)/2}\,\mathrm{d}z \leqslant C|f|_s^2.$$

Proof. The proof is based on the formula (1.11). From the definition of $W_R^s(\mathbb{R}_+)$ it follows that

$$|f \times \mu_r(z)|^2_{s+(n-1)/2} \leqslant C \sum_{k=0}^{\infty} (2k+n)^{2s}|f \times \varphi_k(z)|^2, \tag{3.1}$$

where we have used that fact $k!(n-1)!/(k+n-1)! = 0(k^{-n+1})$. From (3.1) we immediately obtain the theorem.

Using the above theorem together with the embedding theorem of the previous section one can prove the following almost everywhere convergence result for the spherical means.

Theorem 3.2 *If $f \in W_L^s(\mathbb{C}^n)$ with $s > 1/2$ then $w_{2n} f \times \mu_r(z)$ converges to $f(z)$ for almost every z in $\mathbb{C}^n$ as $r \to 0$, where w_{2n} is the measure of the unit sphere in $\mathbb{C}^n$.*

Proof. When $s > 1/2$ the embedding theorem shows that

$$\sup_{r>0}|f \times \mu_r(z)| \leqslant C|f \times \mu_r(z)|_{s+(n-1)/2}$$

and hence one has the inequality

$$\int \sup_{r>0}|f \times \mu_r(z)|^2\,\mathrm{d}z \leqslant C|f|_s^2 \tag{3.2}$$

It is easy to see that $f \times \mu_r(z) \to f(z)$ as $r \to 0$ whenever f is a Schwartz class function. Hence the theorem follows from the above maximal inequality in a routine fashion.

We now turn our attention to the spherical means of L^p functions.

Theorem 3.3 (i) *If $f \in L^1(\mathbb{C}^n)$ and $s < -\frac{1}{2}$ one has*

$$\int |f \times \mu_r(z)|_s^2\,\mathrm{d}z \leqslant C\|f\|_1^2.$$

(ii) *If $f \in L^p(\mathbb{C}^n)$, $1 < p \leqslant 2$ and $s = n(1-1/p) - \frac{1}{2}$,*

$$\int |f \times \mu_r(z)|_s^2\,\mathrm{d}z \leqslant C\|f\|_p^2.$$

Proof. The proof of (i) is rather easy. In fact,

$$|f \times \mu_r(z)|_s^2 \leqslant C \sum_{k=0}^{\infty} (2k+n)^{2s-(n-1)}|f \times \varphi_k(z)|^2$$

and this gives, upon using the estimate

$$\int |f \times \varphi_k(z)|^2 \mathrm{d}z \leqslant C(2k+n)^{n-1} \|f\|_1^2,$$

the inequality

$$\int |f \times \mu_r(z)|_s^2 \mathrm{d}z \leqslant C \|f\|_1^2 \sum_{k=0}^{\infty} (2k+n)^{2s}.$$

The last sum is finite when $s < -\frac{1}{2}$.

In order to prove (ii) we use the mapping properties of the fractional powers L^{-s} of the operator L. Proceeding as above we have

$$\int |f \times \mu_r(z)|_s^2 \mathrm{d}z \leqslant C \sum_{k=0}^{\infty} (2k+n)^{2s-(n-1)} \|f \times \varphi_k\|_2^2 \tag{3.3}$$

and therefore, we only need to show that where $s = n\left(1 - \dfrac{1}{p}\right) - \dfrac{1}{2}$ we have

$$\sum_{k=0}^{\infty} (2k+n)^{-2n(1/p-1/2)} \|f \times \varphi_k\|_2^2 \leqslant C \|f\|_p^2. \tag{3.4}$$

This inequality follows from the more general result given in the next proposition.

PROPOSITION 3.1

If $0 < s < n$ *and* $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{s}{n}$ *where* $1 < p < q < \infty$ *then* $\| L^{-s} f \|_q \leqslant C \|f\|_p$ *for* f *in* $L^p(\mathbb{C}^n)$.

Proof. Let e^{-tL} be the semigroup defined by

$$e^{-tL} f = (2\pi)^{-n} \sum_{k=0}^{\infty} \exp(-(2k+n)t) f \times \varphi_k. \tag{3.5}$$

It is clear that $e^{-tL} f = f \times K_t$ where the kernel K_t is given by

$$K_t(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \exp(-(2k+n)t)\varphi_k(z). \tag{3.6}$$

Using the generating function identity (2.10) we see that

$$K_t(z) = (4\pi)^{-n}(\sinh t)^{-n} \exp(-(1/4)\coth t|z|^2. \tag{3.7}$$

In view of the formula

$$L^{-s} f = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-tL} f \,\mathrm{d}t \tag{3.8}$$

the kernel of L^{-s} is given by

$$K(z) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} K_t(z) \mathrm{d}t. \tag{3.9}$$

Using (3.7) in (3.9) it is now an easy matter to prove the estimate

$$|K(z)| \leqslant C|z|^{-2n+2s}.$$

The proposition now follows from the Hardy-Littlewood-Sobolev theorem for fractional integration (see Stein [9]).

There is an analogue of Theorem 3.3 for the twisted Hardy space $\mathscr{H}^1$. This space was introduced by Mauceri-Picardello-Ricci in [3] and is defined using twisted convolution instead of ordinary convolution. It can also be defined to the subspace of $L^1(\mathbb{C}^n)$ containing all functions f for which the maximal function

$$f^*(z) = \sup_{t>0} |e^{-tL} f(z)| \tag{3.10}$$

is also in $L^1(\mathbb{C}^n)$. There is also an atomic decomposition. Any f in $\mathscr{H}^1$ can be written as

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) \tag{3.11}$$

with $C_1 \|f\|_{\mathscr{H}^1} \leqslant \sum_{k=1}^{\infty} |\lambda_k| \leqslant C_2 \|f\|_{\mathscr{H}^1}$ were the atoms f_k satisfy the properties

(i) f_k is supported in a cube $Q(z_k, r_k)$ centred at z_k and half side r_k
(ii) $\|f_k\|_\infty \leqslant (2r_k)^{-2n}$
(iii) $\int f_k(w) \exp\left(-\frac{i}{2} \operatorname{Im} z_k \cdot \bar{w}\right) dw = 0.$

Using this atomic decomposition we can establish the following result.

Theorem 3.4 *If $f \in \mathscr{H}^1$, $f \times \mu_r(z)$ belongs to $W_R^{-(1/2)}(\mathbb{R}_+)$ for almost every z and one has*

$$\int |f \times \mu_r(z)|^2_{-(1/2)} dz \leqslant C \|f\|^2_{\mathscr{H}^1}.$$

Proof. We need to prove the inequality

$$\sum_{k=0}^{\infty} (2k+n)^{-n} \|f \times \varphi_k\|_2^2 \leqslant C \|f\|^2_{\mathscr{H}^1}. \tag{3.12}$$

It is enough to prove this when f is an atom. So, let f be supported in $Q(z_0, r)$ and satisfying the above conditions (i), (ii) and (iii).

Using the mean zero condition

$$\int f(w) \exp(-(i/2) \mathrm{Im} z.\bar{w}) \, dw = 0 \tag{3.13}$$

we rewrite the twisted convolution $f \times \varphi_k(z)$ as

$$f \times \varphi_k(z) = \int f(w) \Big(\varphi_k(z-w) \exp(-(i/2) \mathrm{Im}\, z.\bar{w}) - \varphi_k(z - z_0) \exp(-(i/2) \mathrm{Im}\, z_0(\bar{w} - \bar{z})) \Big) dw \tag{3.14}$$

An application of Minkowski's integral inequality gives the estimate

$$\|f \times \varphi_k\|_2 \leqslant \int |f(w)| \, \|g_k(.,w)\|_2 \mathrm{d}w, \tag{3.15}$$

where $g_k(z,w)$ is the function inside the bracket on the right hand side of (3.14).

Now a simple calculation shows that

$$\begin{aligned} \|g_k(.,w)\|_2^2 = & \int (\varphi_k(z-w))^2 \mathrm{d}z + \int (\varphi_k(z-z_0))^2 \mathrm{d}z \\ & - \int \varphi_k(z-w)\varphi_k(z-z_0)\exp(-(i/2)\mathrm{Im}(z\cdot\bar{w} - z_0\cdot\bar{w} - z\cdot\bar{z}_0))\mathrm{d}z \\ & - \int \varphi_k(z-w)\varphi_k(z-z_0)\exp(i/2\,\mathrm{Im}(z\cdot\bar{w} - z_0\cdot\bar{w} - z\cdot\bar{z}_0))\mathrm{d}z. \end{aligned} \tag{3.16}$$

The first two terms are equal to $(2\pi)^n\varphi_k(0)$ each and the third term is

$$\begin{aligned} & \int \varphi_k(z+z_0-w)\varphi_k(z)\exp(i/2\,\mathrm{Im}(w-z_0))\cdot\bar{z}\,\mathrm{d}z \\ & \quad = \int \varphi_k(w-z_0-z)\varphi_k(z)\exp\left(\frac{i}{2}\mathrm{Im}(w-z_0)\cdot\bar{z}\right)\mathrm{d}z \\ & \quad = \varphi_k \times \varphi_k(w-z_0) = (2\pi)^n\varphi_k(w-z_0). \end{aligned}$$

Similarly, the fourth term is also $(2\pi)^n\varphi_k(w-z_0)$. Therefore,

$$\int |g_k(z,w)|^2 \mathrm{d}z = (2\pi)^n(\varphi_k(0) - \varphi_k(w-z_0)). \tag{3.17}$$

Writing $\varphi_k(0) = \varphi_k(z_0 - z_0)$ the mean value theorem gives

$$|\varphi_k(0) - \varphi_k(w-z_0)| \leqslant \sum_{j=1}^{n} \sup\left|\frac{\partial}{\partial w_j}\varphi_k\right| |z_0 - w|^2. \tag{3.18}$$

Recall that the Laguerre polynomials $L_k^{n-1}(t)$ satisfy the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}L_k^{n-1}(t) = -\,L_{k-1}^{n}(t) \tag{3.19}$$

and $L_k^n(t)\exp(-(1/2)t)$ satisfy the estimate

$$\sup_{t>0}|\,L_k^n(t)\exp(-(1/2)t)| \leqslant Ck^n. \tag{3.20}$$

Using (3.19) and (3.20) in (3.18) we obtain

$$|\varphi_k(0) - \varphi_k(w-z_0)| \leqslant Ck^n|z_0 - w|^2. \tag{3.21}$$

As f is supported in $|z_0 - w| \leqslant 2r$ we have from (3.15) and (3.21)

$$\|f \times \varphi_k\|_2 \leqslant Ck^{n/2} \int_{|z_0 - w| \leqslant 2r} |f(w)|\,|z_0 - w|\,\mathrm{d}w \leqslant C(2k+n)^{n/2} r. \tag{3.22}$$

Now we are in a position to estimate the sum (3.12). Using (3.22) the sum taken for $k \leqslant r^{-2}$ gives

$$\sum_{k \leqslant r^{-2}} (2k+n)^{-n} \|f \times \varphi_k\|_2^2 \leqslant C. \tag{3.23}$$

On the other hand using the estimate

$$\|f\|_2^2 = \int_{|z_0 - w| \leqslant 2r} |f(w)|\,\mathrm{d}w \leqslant Cr^{-2n} \tag{3.24}$$

we also get

$$\sum_{k > r^{-2}} (2k+n)^{-n} \|f \times \varphi_k\|_2^2 \leqslant Cr^{2n} \|f\|_2^2 \leqslant C. \tag{3.25}$$

Hence (3.12) is established and consequently the theorem follows.

4. A localisation theorem for special Hermite expansions

In this section we prove Theorem B stated in the introduction. The proof is based on the following fact. If a function $g \in L^2(\mathbb{R}_+, r^{2n-1}\mathrm{d}r)$ then the Fourier-Laguerre coefficients $(g, \psi_k) \to 0$ as $k \to \infty$. Recalling the definition of ψ_k, this means that

$$\int_0^\infty g(r)\varphi_k(r) r^{2n-1}\,\mathrm{d}r = \mathrm{o}((2k+n)(n-1)/2) \tag{4.1}$$

as $k \to \infty$. If in addition $g \in W_R^s(\mathbb{R}_+)$ then it follows that

$$\int_0^\infty g(r)\varphi_k(r) r^{2n-1}\,\mathrm{d}r = \mathrm{o}((2k+n)(n-1)/2). \tag{4.2}$$

In view of Theorem 3.1, Theorem B is an immediate corollary of the following result.

Theorem 4.1. *Let $n \geqslant 2$ and f be a compactly supported function vanishing in a neighbourhood of a point $z \in \mathbb{C}^n$. Further assume that $f \times \mu_r(z)$ belongs to $W_R^{n/2}(\mathbb{R}_+)$ as a function of r. Then $S_N f(z) \to 0$ as $N \to \infty$.*

Proof. From the relation (1.11) and the orthogonality of φ_k we can write the partial sums $S_N f(z)$ as

$$S_N f(z) = \int_0^\infty f \times \mu_r(z) \left(\sum_{k=0}^{N} \varphi_k(r) \right) r^{2n-1}\,\mathrm{d}r. \tag{4.3}$$

Using the formulas (see Szego [12])

$$\sum_{k=0}^{N} L_k^{n-1}\left(\frac{1}{2}r^2\right) = L_N^n\left(\frac{1}{2}r^2\right), \tag{4.4}$$

$$\frac{1}{2}r^2 L_N^n\left(\frac{1}{2}r^2\right) = (N+n)\, L_N^{n-1}\left(\frac{1}{2}r^2\right) - (N+1)\, L_{N+1}^{n-1}\left(\frac{1}{2}r^2\right) \tag{4.5}$$

we can write (4.3) in the form

$$S_N f(z) = \int_0^\infty g_z(r)((N+n)\varphi_N(r) - (N+1)\varphi_{N+1}(r))r^{2n-1}\,dr \tag{4.6}$$

where we have written $g_z(r) = 2r^{-2} f \times \mu_r(z)$. Rewriting

$$(N+n)\, L_N^{n-1}\left(\frac{1}{2}r^2\right) - (N+1)\, L_{N+1}^{n-1}\left(\frac{1}{2}r^2\right)$$
$$= (N+n)\left(L_N^{n-1}\left(\frac{1}{2}r^2\right) - L_{N+1}^{n-1}\left(\frac{1}{2}r^2\right)\right) + (n-1)\, L_{N+1}^{n-1}\left(\frac{1}{2}r^2\right) \tag{4.7}$$

and using the formula

$$L_{N+1}^{n-1}\left(\frac{1}{2}r^2\right) - L_N^{n-1}\left(\frac{1}{2}r^2\right) = L_{N+1}^{n-2}\left(\frac{1}{2}r^2\right) \tag{4.8}$$

we finally express (4.6) as the sum of the following two terms:

$$(n-1)\int_0^\infty g_z(r)\varphi_{N+1}(r)r^{2n-1}\,dr, \tag{4.9}$$

and

$$(N+n)\int_0^\infty g_z(r)\, L_{N+1}^{n-2}\left(\frac{1}{2}r^2\right)\exp(-(1/4)r^2)r^{2n-1}\,dr. \tag{4.10}$$

Now if f is supported in $|w| \leqslant b$ and vanishes in $|z-w| \leqslant a$ then $f \times \mu_r(z)$ as a function of r is supported in $a \leqslant r \leqslant b + |z|$. Consider the integral (4.9) first. As we are assuming that $f \times \mu_r(z)$ belongs to $W_R^{n/2}(\mathbb{R}_+)$ it also belongs to $H^{n/2}(\mathbb{R}_+)$. As it vanishes near $0, g_z(r)$ belongs to $H^{n/2}(\mathbb{R}_+)$ as well. The compactness of the support of g_z implies that $g_z \in W_R^{n/2}(\mathbb{R}_+)$ and hence by the remark (4.2) it follows that

$$\int_0^\infty g_z(r)\varphi_{N+1}(r)r^{2n-1}\,dr = o((2N+n)^{-(1/2)}). \tag{4.11}$$

Thus the term (4.9) tends to 0 as $N \to \infty$.

We write the term (4.10) as

$$(N+n)\int_0^\infty f \times \mu_r(z)\, L_{N+1}^{n-2}\left(\frac{1}{2}r^2\right)\exp(-(1/4)r^2)r^{2n-3}\,dr. \tag{4.12}$$

The function $f \times \mu_r(z)$ belongs to $W_R^{n/2}(\mathbb{R}_+)$ and as it is compactly supported it also belongs to the same space defined using $L_k^{n-2}(1/2r^2)\exp(-(1/4)r^2)$ instead of $\varphi_k(r)$. Therefore,

$$\int_0^\infty f \times \mu_r(z) L_{N+1}^{n-2}\left(\frac{1}{2}r^2\right)\exp(-(1/4)r^2)r^{2n-3}\mathrm{d}r = o(N^{-1}) \tag{4.13}$$

and hence the term (4.10) also converges to 0 as $N \to \infty$. This completes the proof of the theorem.

In the case of compact symmetric spaces of rank one a localisation theorem for spherical harmonic expansions was proved under the only assumption that f is in L^2. As we are dealing with the noncompact situation in order to prove theorem B we have had to assume a further regularity assumption on f. We don't know if the condition $f \in W_L^{1/2}(\mathbb{C}^n)$ is optimal; nevertheless some regularity assumption is essential as the following counter-example shows.

Consider the function f which is the characteristic function of the annulus $1 \leqslant |z| \leqslant 2$. As f is radial, the special Hermite expansion takes the form

$$f(z) = c_n \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!}\left(\int_1^2 \varphi_k(s)s^{2n-1}\mathrm{d}s\right)\varphi_k(z), \tag{4.14}$$

where c_n is a constant. Hence the spherical means $f \times \mu_r(z)$ has the expansion

$$f \times \mu_r(z) = c_n' \sum_{k=0}^{\infty}\left(\frac{k!(n-1)!}{(k+n-1)!}\right)^2\left(\int_1^2 \varphi_k(s)s^{2n-1}\mathrm{d}s\right)\varphi_k(r)\varphi_k(z). \tag{4.15}$$

Setting $z = 0$ and remembering $\varphi_k(0) = (k+n-1)!/k!(n-1)!$ we get

$$f \times \mu_r(0) = c_n' \sum_{k=0}^{\infty}\left(\frac{k!(n-1)!}{(k+n-1)!}\right)^{1/2}\left(\int_1^2 \varphi_k(s)s^{2n-1}\mathrm{d}s\right)\psi_k(r). \tag{4.16}$$

It is clear that $f \times \mu_r(0)$ belongs to $L^2(\mathbb{R}_+, r^{2n-1}\mathrm{d}s)$. We claim that it does not belong to $W_R^{n/2}(\mathbb{R}_+)$ and the partial sums $S_N f(0)$ does not converge to 0.

In order to prove these two claims we need the following asymptotic property of the Laguerre functions (see Szego [12]): for $x > 0$

$$L_k^\alpha(x^2)\exp(-(1/2)x^2)x^\alpha = k^{-\alpha/2}\left(\frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}\right)J_\alpha(2\sqrt{K}x) + 0(x^{\alpha/2-3/4}) \tag{4.17}$$

where $K = k + (\alpha+1)/2$ and the bound holds uniformly in $0 < x \leqslant x_0$. As $J_\alpha(t)$ behaves like $t^{-1/2}$ as $t \to \infty$ it follows from (4.17) that

$$\varphi_k(s) \sim (2k+n)^{(n-1)/2-1/4}, 1 \leqslant s \leqslant 2 \tag{4.18}$$

as $k \to \infty$. This shows that $f \times \mu_r(0)$ is certainly not in $W_R^{n/2}(\mathbb{R}_+)$.

To see that $S_N f(0)$ does not converge we have

$$S_N f(0) = \int_0^\infty f \times \mu_r(0)\, L_N^n\left(\frac{1}{2}r^2\right)\exp(-(1/4)r^2)r^{2n-1}\mathrm{d}r. \tag{4.19}$$

A calculation shows that $f \times \mu_r(0) = 1$ for $1 \leqslant r \leqslant 2$ and is zero elsewhere. Thus

$$S_N f(0) = \int_1^2 L_N^n\left(\frac{1}{2}r^2\right)\exp(-(1/4)r^2)r^{2n-1}\,\mathrm{d}r. \tag{4.20}$$

Again (4.17) shows that $S_N f(0)$ behaves like $N^{n/2-1/4}$ as $N \to \infty$ and hence $S_N f(0)$ does not converge to 0 as $N \to \infty$.

In a recent article [5] Mark Pinsky has related the smoothness of the spherical means with pointwise convergence of the partial seems associated to Fourier integrals and Fourier series. In a similar vein we can also prove pointwise convergence of special Hermite expansions under some smoothness properties of the twisted spherical means. For example, let $n = 1$ and consider the partial sums $S_N f(z)$ which is given in terms of the spherical means as

$$S_N f(z) = \int_0^\infty f \times \mu_r(z)\, L_N^1\left(\frac{1}{2}r^2\right)\exp(-(1/4)r^2)r\,\mathrm{d}r. \tag{4.21}$$

Assume that $f \times \mu_r(z)$ is supported in $0 < r \leqslant a$ and that $\mathrm{d}/\mathrm{d}r\, f \times \mu_r(z)$ is piecewise continuous, the discontinuties being at $a_1 < a_2 < \ldots < a_m$ where $a_1 > 0$ and $a_m < a$.

Using the relation $\mathrm{d}/\mathrm{d}r\, L_k^\alpha(t) = -L_{k-1}^{\alpha+1}(t)$ we may write (4.21) as

$$S_N f(z) = \int_0^a f \times \mu_r(z)\exp(-(1/4)r^2)\frac{\mathrm{d}}{\mathrm{d}r}\left\{L_{N+1}^0\left(\frac{1}{2}r^2\right)\right\}\mathrm{d}r. \tag{4.22}$$

Integrating by parts we get

$$S_N f(z) = f(z+0) + \int_0^a \frac{\mathrm{d}}{\mathrm{d}r}\{f \times \mu_r(z)\exp(-(1/4)r^2)\}\, L_{N+1}^0\left(\frac{1}{2}r^2\right)\mathrm{d}r, \tag{4.23}$$

where $f(z+0) = \lim_{r\to 0} f \times \mu_r(z)$. By the asymptotic properties of L_N^0 it is easily seen that the integral on the right hand side of (4.23) tends to 0 as $N \to \infty$. Thus $S_N f(z) \to f(z+0)$ as $N \to \infty$. More generally, we can prove the following theorem.

Theorem 4.2. *Assume that $f \times \mu_r(z)$ is compactly supported in r and $\mathrm{d}^n/\mathrm{d}r^n(f \times \mu_r(z))$ is piecewise continuous. Then $S_N f(z) \to f(z+0)$ as $N \to \infty$.*

Proof. The proof follows by iteration. We have

$$S_N f(z) = \int_0^a f \times \mu_r(z)\exp(-(1/4)r^2)\, r^{2n-1}\, L_N^n\left(\frac{1}{2}r^2\right)\mathrm{d}r. \tag{4.24}$$

If D stands for the operator $1/r\, \mathrm{d}/\mathrm{d}r$ then we have the relation

$$(-1)^n D^n\left(L_{N+n}^0\left(\frac{1}{2}r^2\right)\right) = L_N^n\left(\frac{1}{2}r^2\right). \tag{4.25}$$

Using this to integrate by parts in (4.24) and remembering that $D^{n-1}(f \times \mu_r(z))$ is a continuous function of r we can prove that $S_N f(z) \to f(z+0)$ as $N \to \infty$ as in the case of $n = 1$. The details are left to the reader.

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Weakly wandering sets and compressibility in descriptive setting

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Abstract. A Borel automorphism T on a standard Borel space $(X, \mathbb{B})$ is constructed such that (a) there is no probability measure invariant under T and (b) there is no Borel set W weakly wandering under T and which generates the invariant set X.

Keywords. Borel automorphism; weakly wandering set; compressibility.

1. Introduction

The purpose of this note is to give an example of a Borel automorphism T on a standard Borel space $(X, \mathscr{B})$ such that (i) T does not admit any invariant probability measure, (ii) there is no weakly wandering set $W \in \mathscr{B}$ such that union of $T^n W$ over all integers n is all of X. This shows that a natural descriptive version of the Hajian-Kakutani theorem on the existence of a finite equivalent invariant measure (explained below) is false. This answers a question raised in [10].

Let m be a probability measure on $\mathscr{B}$ and assume that null sets of m are preserved by T. Call a set $W \in \mathscr{B}$ weakly wandering if there exists a sequence of integers (n_i), $i = 1, 2, 3, \ldots$, such that $T^{n_i} W \cap T^{n_j} W = \emptyset$ whenever $i \neq j$. It is easy to see that if there is a weakly wandering set of positive m-measure then there is no probability measure on $\mathscr{B}$ invariant under T and having same null sets as m. A well known theorem of Hajian and Kakutani (see [4], [3]) states that the converse of this observation holds, i.e., if there is no weakly wandering set of positive m measure then there exists a probability measure on $\mathscr{B}$ invariant under T and having the same null sets as m. This result was further improved by Jones and Krengel [7] who showed that if there is no T-invariant probability measure on $\mathscr{B}$ absolutely continuous with respect to m then there exists a set W in $\mathscr{B}$ and a sequence of positive integers (n_i), $i = 1, 2, 3, \ldots$ such that $T^{n_i} W \cap T^{n_j} W = \emptyset$ whenever $i \neq j$ and $\cup_{i=1}^{\infty} T^{n_i} W = X \pmod m$. (The qualification mod m cannot in general be removed as the result of this paper will show. However, as pointed out to us by U Krengel, if we take the completion $\mathscr{M}$ of the σ-algebra $\mathscr{B}$ with respect to m then the requirement mod m can be dispensed with. (See remark at the end of § 2.)

Let us dispense with the measure m and consider only the standard Borel space $(X, \mathscr{B})$ and the Borel automorphism T on X. Say that a set $A \in \mathscr{B}$ is of full saturation if $\cup_{n=-\infty}^{\infty} T^n A = X$. We note that if T admits a weakly wandering set of full saturation then T does not admit a T-invariant probability measure. It is natural to ask, in analogy with the Hajian-Kakutani theorem mentioned above, whether the non-existence of a weakly wandering set of full saturation implies the existence of a T-invariant probability measure of $\mathscr{B}$. As stated in the beginning, the answer to this question is in the negative.

2. The main construction

DEFINITION

Given a non-singular system $(X, \mathscr{B}, m, T)$ a sequence of natural numbers $n_1, n_2, n_3, \ldots$ is said to be weakly wandering for T if there exists a set W in $\mathscr{B}$ of positive m measure such that $T^{n_k} W, k = 1, 2, 3, \ldots$ are pairwise disjoint.

DEFINITION

A sequence $r_1, r_2, r_3, \ldots$ of natural numbers is said to be recurrent for T if its intersection with every weakly wandering sequence for T is finite.

Clearly a recurrent sequence for T cannot be a weakly wandering sequence for T and vice-versa. The notion of weakly wandering, recurrent and other related sequences were investigated by Hajian and Ito [5].

Theorem 1. *Let T be an ergodic measure preserving transformation on a σ-finite measure space $(X, \mathscr{B}, m)$. A sequence $r_1, r_2, r_3, \ldots$ of natural numbers is recurrent for T if and only if there is a set A of positive finite measure such that* $\overline{\lim_{n \to \infty}} (T^{r_n} A \cap A) > 0$.

For a proof of the above theorem we need the following two lemmas. We state and prove them for the sake of completeness; see [5].

Lemma 1. *Let $W = \{w_i\}$ be a weakly wandering sequence for the ergodic measure preserving transformation T. Then $\lim_{i \to \infty} m(T^{w_i} A \cap A) = 0$ for every set $A \in \mathscr{B}$ with $m(A) < \infty$.*

Proof. Let A be a set with $m(A) < \infty$, and let $W \in \mathscr{B}$ be a weakly wandering set of positive measure under the sequence $\mathbf{W} = \{w_i\}$ for T. For any integer $k > 0$ it is clear that $T^k W$ is again weakly wandering under both $\{w_i\}$ and $\{-w_i\}$. Since

$$m(A) \geqslant m\left(\bigcup_{i=1}^{\infty} T^{-w_i}(T^k W) \cap A\right) = \sum_{i=1}^{\infty} m(T^{-w_i}(T^k W) \cap A) =$$
$$\sum_{i=1}^{\infty} m(T^k W \cap T^{w_i} A),$$

it follows that

$$\lim_{i \to \infty} m(T^{w_i} A \cap T^k W) = 0 \text{ for any integer } k > 0.$$

Since $m(A) < \infty$ and T is ergodic it follows that for any $\varepsilon > 0$ there exists an integer $N > 0$ and a set C with $m(C) < \varepsilon$ such that

$$A \subset C \cup \bigcup_{j=0}^{N} T^j W.$$

Then

$$m(T^{w_i}A\cap A)\leqslant m(T^{w_i}A\cap C)+\sum_{k=0}^{N} m(T^{w_i}A\cap T^k W),$$

and this implies

$$\overline{\lim}_{i\to\infty} m(T^{w_i}A\cap A)<\varepsilon.$$

Lemma 2. *Let* $\{n_i\}$ *be a sequence of integers such that* $\lim_{i\to\infty} m(T^{n_i}A\cap A)=0$ *for every set* $A\in\mathscr{B}$ *with* $m(A)<\infty$. *Then* $\{n_i\}$ *contains a weakly wandering subsequence.*

Proof. Let C and D be two sets of finite measure. Since $\lim_{i\to\infty} m(T^{n_i}A\cap A)=0$ for every set of finite measure and $m[T^{n_i}(C\cup D)\cap(C\cup D)]\geqslant m(T^{n_i}C\cap D)$ it follows that $\lim_{i\to\infty} m(T^{n_i}C\cap D)=0$ for any two sets C and D of finite measure.

Let C be a set with $0<m(C)<\infty$, and let $\varepsilon>0$ be such that $\varepsilon<m(C)$. We show that C contains a subset $W\subset C$, with $m(C-W)<\varepsilon$, and such that W is weakly wandering under a subsequence $\{w_i\}$ of the sequence $\{n_i\}$.

Put $w_0=0$ and choose $\varepsilon_i>0$ for $i=1,2,\ldots$ such that $\Sigma\varepsilon_i=\varepsilon$. Since $\lim_{i\to\infty} m(T^{n_i}C\cap C)=0$ we choose $w_1\in\{n_i\}$ such that $m(T^{w_1}C\cap C)<\varepsilon_1$. We let $D=C\cup T^{-w_1}C$, and since $\lim_{i\to\infty} m(T^{n_i}D\cap C)=0$ we choose $w_2\in\{n_i\}$ such that $m([T^{w_2}C\cup T^{w_2-w_1}C]\cap C)<\varepsilon_2$. We proceed inductively; having chosen the integers $w_0,w_1,\ldots,w_k$ satisfying

$$m\left[\left(\bigcup_{j=0}^{i-1} T^{w_i-w_j}C\right)\cap C\right]<\varepsilon_i,\quad 1\leqslant i\leqslant k.$$

We put $D=\bigcup_{j=0}^{k} T^{-w_j}C$ and choose $w_{k+1}\in\{n_i\}$ such that $m(T^{w_{k+1}}D\cap C)<\varepsilon_{k+1}$ or

$$m\left[\left(\bigcup_{j=0}^{k} T^{w_{k+1}-w_j}C\right)\cap C\right]<\varepsilon_{k+1}.$$

Next we let

$$C'=\bigcup_{i=1}^{\infty}\bigcup_{j=0}^{i-1} T^{w_i-w_j}C\cap C.$$

It follows that the set $W=C-C'$ has positive measure, $m(C-W)<\varepsilon$ and since $W\subset C$ and $T^{w_i-w_j}W\subset T^{w_i-w_j}C\subseteq C'\cup(X-C)$ for $i>j$. We conclude that $T^{w_i}W\cap T^{w_j}W=\varnothing$ for $i\neq j$. Thus $\{w_i\}$ is a subsequence of $\{n_i\}$ which is weakly wandering for T. Q.E.D.

Proof of Theorem 1. Since a subset of a weakly wandering sequence is again weakly wandering, the sufficiency of the condition follows from Lemma 1; the necessity follows from Lemma 2. Q.E.D.

Let $I=[0,1)$ and let V denote the von Neumann transformation (also known as the adding machine) acting on I. If $x=x_1x_2x_3\ldots$ is the binary expansion of $x\in I$ then $V(x)=000\ldots01x_{k+2}x_{k+3}\ldots$ where $k+1$ is the first integer i for which x_i is zero. V is unambiguously defined except for countably many points which admit two distinct binary expansions. V preserves Lebesgue measure on $[0,1)$ and is uniquely ergodic.

For each sequence $\alpha=(k_1,k_2,k_3,\dots)$ on non-negative integers construct a tower over V with ceiling function

$$f_\alpha(x)=\begin{cases}0 & \text{if } 0\leqslant x<1/2\\ k_1 & \text{if } 1/2\leqslant x<1/2+1/4\\ k_1+k_2 & \text{if } 1/2+1/4\leqslant x<1/2+1/4+1/8\\ & \cdot\\ & \cdot\\ & \cdot\end{cases}$$

In other words let

$$X_\alpha=\{(x,n):x\in[0,1),0\leqslant n<f_\alpha(x)\}$$

equipped with the natural Borel structure $\mathscr{B}_\alpha$ as a subset of $[0,1)\times\{0,1,2,3,\dots\}$. Let m_α be the measure on $\mathscr{B}_\alpha$ which agrees with m on $[0,1)\times\{0\}$ and which is invariant under T_α defined on X_α by:

$$T_\alpha(x,n)=\begin{cases}(x,n+1) & \text{if } 0\leqslant n<f_\alpha(x)\\ (Vx,0) & \text{if } n=f_\alpha(x)\end{cases}.$$

The map T_α can also be viewed as a rank one transformation where at stage n of the construction the stack obtained at stage $(n-1)$ is divided into two equal columns and we add $k_1+k_2+k_3+\cdots+k_n$ spacers on the left hand column. Observe that if J is one of the intervals in the (n)th stack of height h_n and if J is divided into two equal intervals J_1 and J_2, J_1 to the left of J_2, then $T_\alpha^{r_n}J_1=J_2$ where $r_n=h_n+k_1+\cdots+k_{n+1}$. This at once shows that $m(T_\alpha^{r_n}I\cap I)\geqslant 1/2$. (Observe that I is partitioned into disjoint intervals in the nth stack. The left half of each of these intervals coincides with the right half on application of $T_\alpha^{r_n}$.) By a slight refinement of this argument, if one chooses the sequence $\{k_n\}$ carefully, it is possible to construct an ergodic measure preserving transformation T which is of type 1/2. By a transformation of type $\alpha, 0\leqslant\alpha\leqslant 1$, we mean an ergodic measure preserving transformation T with the property that $\limsup_{n\to\infty}m(T^nA\cap A)=\alpha m(A)$ for every set $A\in\mathscr{B}$ with $m(A)<\infty$. We do not need this additional refinement.

Theorem 2. *Given an increasing sequence of natural numbers $\{n_i\}$ there exists an increasing sequence $\alpha=\{k_n\}$ of natural numbers such that for the constructed space $m(X_\alpha)=\infty$ and further there exists a subsequence $\{r_i\}$ of $\{n_i\}$ which is recurrent for T_α.*

Proof. Let us suppose that an increasing sequence of positive integers $\{n_i\}$ is given. We choose inductively a subsequence $\{r_i\}$ of $\{n_i\}$ and construct a sequence $\{k_i\}$ of non-negative integers as follows. We choose $r_1\in\{n_i\}$ such that $r_1>0$ and let $k_1=r_1-1$. Next we choose $r_2\in\{n_i\}$ such that $r_2>4r_1$ and let $k_2=r_2-r_1-k_1$. We continue this way; having chosen the integers $r_1,r_2,\cdots,r_{i-1}$ and constructed the integers $k_1,k_2,\cdots,k_{i-1}$, we choose $r_i\in\{n_j\}$ such that $r_i>4(r_1+r_2+\cdots+r_{i-1})$ and let $k_i=r_i-(k_1+k_2+\cdots+k_{i-1})-r_{i-1}$. We note that the sequence of integers $\{k_i\}$ thus constructed satisfies $\Sigma k_n/2^n=\infty$. Thus, if we build the transformation T_α with $\alpha=\{k_n\}$ just chosen, then it follows from the above construction that $m(X_\alpha)=\infty$ and $m(T_\alpha^{r_i}I\cup I)\geqslant 1/2$ for all i. By Theorem 1 $\{r_i\}$ is a recurrent sequence for T_α. Q.E.D.

We now give an example of a Borel automorphism T having properties (i) and (ii) stated in the beginning.

Let R denote all sequences of natural numbers equipped with its product Borel structure. The set of points in αR for which m_α is finite is given by $F = \{(k_1, k_2, k_3, \ldots): \Sigma k_i 2^{-i} < \infty\}$ which is clearly a Borel subset of R. Let $E = R - F$. Let $\Omega = E \times [0, 1) \times \{0, 1, 2, 3, \ldots\}$ equipped with the product σ-algebra and set

$$X = \{(\alpha, x, n) \in \Omega : n \leqslant f_\alpha(x)\}.$$

Then X is a Borel subset of Ω since $(\alpha, x) \to f_\alpha(x)$ is jointly measurable on $E \times [0, 1)$. Define T on X by requiring it to be T_α on the α section of X

$$T(\alpha, x, n) = (\alpha,\ T_\alpha(x, n)).$$

T is Borel measurable and leaves each α section invariant.

Clearly T does not admit a weakly wandering set of full saturation since for any sequence of natural numbers there exists a T_α which does not admit the chosen sequence as a weakly wandering sequence. Also T does not admit a T-invariant probability measure, for if it did then the regular conditional probabilities (see Parthasarathy [11]) of this measure with respect to the measurable partition given by the α sections of X would be invariant with respect to T_α for almost every α. But no T_α admits an invariant probability measure on $\mathscr{B}_\alpha$ since α is in E and V is uniquely ergodic. This contradiction shows that T does not admit an invariant probability measure.

Remark. As mentioned in the introduction, if we replace $\mathscr{B}$ by $\mathscr{M}$ the completion of $\mathscr{B}$ with respect to a finite or σ-finite measure μ then it is possible to find a set $W \in \mathscr{M}$ and an increasing sequence of positive integers $n_1, n_2, \ldots$ such that $\bigcup_{i=0}^{\infty} T^{n_i}(W) = X$. We see this as follows. Since there is no invariant probability measure for T, by the theorem of Jones and Krengel mentioned in the introduction we can find a Borel set A and a sequence of positive integers $n_1 < n_2 < \ldots$ such that the union of iterates of A under this sequence is an invariant set Y with $\mu(X - Y) = 0$. Moreover the sequence $n_1 < n_2 < \ldots$ can be so chosen that for a suitable subset E of integers the union

$$\bigcup_{i=0}^{\infty} (E + n_i)$$

is the set of all integers. Since we are in completed σ-algebra every subset of $X - Y$ is in $\mathscr{M}$. We can find set B, if necessary by use of axiom of choice, such that orbit of every point in $X - Y$ intersects B in exactly one point. Let $C = \cup_{i \in E} T^i(B)$. The union of A and C is the required set W.

3. Hopf's theorem and related perspective

An old expository paper of Birkhoff and Smith [2] sets forth nearly all the basic notions of classical ergodic theory. The question of existence of a finite invariant measure is discussed in § 4 of this paper. They discuss the question for a continuous invertible mapping of a surface. The two notions, namely of weakly wandering sets and compressibility introduced there have remained important in subsequent discussions of this problem in the measure theoretic setting. Birkhoff and Smith begin by observing that even if a continuous transformation does not admit a non-empty wandering open set W, it may yet admit a non-empty open W whose iterates under T over a subsequence of integers are pairwise disjoint. This forbids the existence of a finite T-invariant measure which gives positive mass to every open set. No concrete example of such a situation is exhibited however. Instead they introduce their notion of compressibility. From this they are able to formulate a necessary and sufficient condition for the existence of a finite invariant measure. Their definition of compressibility was constrained by the requirement that partitions of the space into only finitely many sets were allowed. On the suggestion of Birkhoff, Hopf considered the question of existence of a finite invariant measure for a nonsingular transformation [6]. He modified their definition of compressibility by allowing countable partitions of the space and by analyzing the problem in a measure theoretic setting. We report his work below.

Let $(X, \mathscr{B})$ be a standard Borel space and $T: X \leftrightarrow X$ a Borel automorphism. Two sets A, B in $\mathscr{B}$ are said to be equivalent by countable decomposition, $A \sim B$, if we can write (i) A as a countable union of pairwise disjoint sets A_i in $\mathscr{B}$, $i = 1, 2, 3, \ldots$, (ii) B as a countable union of pairwise disjoint sets $B_i \in \mathscr{B}$, $i = 1, 2, 3, \ldots$ and (iii) there exist integers n_i such that $T^{n_i} A_i = B_i, i = 1, 2, 3, \ldots$. This notion of equivalence by countable decomposition is basic to the theory of orbit equivalence. (Equivalence by countable decomposition of two sets essentially means that the induced transformations on the two sets are orbit equivalent.) Suppose that m is a finite measure on $\mathscr{B}$ whose null sets are preserved by T. We say that a set A in $\mathscr{B}$ is compressible in the sense of Hopf if there exists a set $B \in \mathscr{B}$, $B \subset A$, such that $m(A - B) > 0$ and A and B are equivalent by countable decomposition. It is easy to see that if there is a finite T invariant measure μ having the same null sets as m, then X is not compressible in the sense of Hopf. Hopf proved the non-trivial converse of this. He showed that if X is not compressible in his sense then there is a finite T invariant measure μ on $\mathscr{B}$ whose null sets agree with those of m.

Hopf's proof of his theorem was difficult. A simpler proof of his result and also simpler necessary and sufficient conditions for the existence of a finite equivalent invariant measure were therefore sought. The best known result in this connection is the theorem of Hajian and Kakutani mentioned above. Since there exists ergodic measure preserving transformations on a non-atomic infinite σ-finite measure space, such transformations always admit weakly wandering sets of positive measure. Thus Hajian and Kakutani proved not only the non-trivial converse of the observation due to Birkhoff and Smith but also that weakly wandering sets of positive measure exist in abundance in suitable non-dissipative systems.

There is a simple connection between weakly wandering sets and compressibility in the sense of Hopf. Suppose W in $\mathscr{B}$ is weakly wandering under T and of positive measure, $T^{n_k} W, k = 1, 2, 3, \ldots$ being pairwise disjoint. Then we can compress X into $X - W$ simply by mapping $W \rightarrow T^{n_1} W$, $T^{n_1} W \rightarrow T^{n_2} W \ldots$, $T^{n_k} W \rightarrow T^{n_{k+1}} W, \ldots$ and letting the identity map act on $X - \cup T^{n_k} W$. Thus we see that X is compressible in the sense of Hopf and a simple compression can be effected by means of iterates of a weakly wandering set of positive measure. Therefore, if X is not compressible in the sense of Hopf then T can not admit a weakly wandering set of positive measure and so by the Hajian-Kakutani theorem admits a finite invariant measure having the same null sets as m. Hopf's Theorem follows.

Mackey has emphasized the importance of considering group actions on standard Borel spaces free of any measure on the space. (see [8], [9]). There has been renewed interest in this aspect of classical ergodic theory in recent years (see Baker and Kechris [1] and references therein). In this spirit, Hopf's theorem has the following descriptive formulation proved in Nadkarni [10]: Given a standard Borel space $(X, \mathscr{B})$ and a Borel automorphism T on X call X compressible if we can write X as a disjoint union of two sets C and D in $\mathscr{B}$ such that (a) C and D have the same saturation and (b) X is equivalent by countable decomposition to C. We note that if X is compressible then it is compressible in the sense of Hopf with respect to any probability measure quasi-invariant under T. Also it is easy to see that if X is compressible then there is no T-invariant probability measure on $\mathscr{B}$. The converse of this holds: if X is not compressible then T admits an invariant probability measure on $\mathscr{B}$. This is the descriptive version of Hopf's theorem proved in [10].

It is clear that if X admits a weakly wandering set in $\mathscr{B}$ of full saturation then X is compressible. The example of this paper shows that converse of this is not true: The T constructed above does not admit an invariant probability measure and so X is compressible under T, but T does not admit a weakly wandering set of full saturation.

Our analysis raises the following two problems of some interest: (i) When does a Borel automorphism T on a standard Borel space admit a weakly wandering set of full saturation? (ii) If X is descriptively compressible under T, is there a countably generated partition of X into invariant sets on each of which T admits a weakly wandering set of full saturation?

Remark. The authors would like to take this opportunity to express their appreciation of the long forgotten paper of Birkhoff and Smith [2] which at once sets forth, albeit in slightly weak form, nearly all the basic notions of classical ergodic theory: wandering sets, decomposition into conservative and dissipative parts, weakly wandering sets and compressibility, ergodicity (metric transitivity), equivalence or singularity of two ergodic measures, etc.

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On infinitesimal σ-fields generated by random processes

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Abstract. It is proved that the infinitesimal look-ahead and look-back σ-fields of a random process disagree at atmost countably many time instants.

Keywords. Random processes; look-ahead σ-field; look-back σ-field; Markov processes.

Let $X(t), t \geqslant 0$, be a Polish space-valued random process defined on a probability space $(\Omega, \mathscr{F}, P)$, where $\mathscr{F}$ is the P-completion of a countably generated σ-field $\mathscr{F}_0$. Let $\xi(t+)$ (the 'look-ahead' σ-field at t) and $\xi(t-)$ (the 'look-back' σ-field at t) denote the P-completions of $\cap_{s>t}\sigma(X(y), t \leqslant y \leqslant s)$ and $\cap_{s<t}\sigma(X(y), s \leqslant y \leqslant t)$ respectively. The aim of this note is to prove the following fact:

Theorem 1. $\xi(t+) = \xi(t-)$ *for all but at most countably many* $t \geqslant 0$.

We shall prove this through a sequence of lemmas. Let $\{A_n, n \geqslant 1\}$ be a dense collection of sets in $\mathscr{F}_0$ which generates $\mathscr{F}_0$.

Lemma 1. *The space of all P-complete sub-σ-fields of* $\mathscr{F}$ *forms a metric space under the metric*

$$d(\mathscr{G}_1, \mathscr{G}_2) = \sum_n 2^{-n} E[|P(A_n/\mathscr{G}_1) - P(A_n/\mathscr{G}_2)|].$$

The proof is easy.

Remark. This topology was first introduced in Cotter [3].

For $t \geqslant 0$, let

$$\mathscr{F}(t) = \sigma(X(s), 0 \leqslant s \leqslant t), \quad \mathscr{F}(t+) = \bigcap_{s>t} \mathscr{F}(s), \quad \mathscr{F}(t-) = \bigvee_{s<t} \mathscr{F}(s),$$

all completed with respect to P. For $n \geqslant 1$, $t \geqslant 0$, define

$$h_n(t) = E[(E[I_{A_n}/\mathscr{F}(t)])^2].$$

Lemma 2. (i) $t \to h_n(t)$ *is bounded nondecreasing for all n.*
(ii) $\lim_{s\downarrow\downarrow t} h_n(s) = E[(E[I_{A_n}/\mathscr{F}(t+)])^2]$.
(iii) $\lim_{s\uparrow\uparrow t} h_n(s) = E[(E[I_{A_n}/\mathscr{F}(t-)])^2]$.

Proof. (i) follows from the conditional Jensen's inequality and (ii), (iii) follow from the convergence theorems for regular martingales and reversed martingales [2]. □

From (i) above, it follows that each $h_n(\cdot)$ has at most a countable set of points of discontinuity. Let $D \subset [0, \infty)$ be the at most countable set of points where one or more of the $h_n(\cdot)$'s is discontinuous.

Lemma 3. *For* $t \notin D, \mathscr{F}(t+) = \mathscr{F}(t-)$.

Proof. For $t \notin D$, Lemma 2 (ii), (iii) imply that

$$E[(E[I_{A_n}/\mathscr{F}(t+)])^2] = E[(E[I_{A_n}/\mathscr{F}(t-)])^2], \quad n \geqslant 1.$$

Thus

$$E[(E[I_{A_n}/\mathscr{F}(t+)] - E[(E[I_{A_n}/\mathscr{F}(t-)])^2] = 0, \quad n \geqslant 1,$$

implying $E[I_{A_n}/\mathscr{F}(t+)] = E[I_{A_n}/\mathscr{F}(t-)]$ a.s., $n \geqslant 1$. The claim follows from Lemma 1. □

COROLLARY 1

$\xi(t+) \subset \xi(t-)$ *for all but at most countably many* t.

Proof. Let $\{r_m\}$ be an enumeration of rationals in $[0, \infty)$. Define $\mathscr{F}^m(t)$, $\mathscr{F}^m(t+)$, $\mathscr{F}^m(t-)$, $h_n^m(t)$ as in (1)–(4) resp. with $X(r_m + .)$ replacing $X(\cdot)$, $m \geqslant 1$. The foregoing results hold for each $X(r_m + .)$ as well. For every rational $r \geqslant 0$, let

$$D_r = \{t > r \mid P\text{-completion of } \bigvee_{\varepsilon>0} \sigma(X_s, r \leqslant s \leqslant t - \varepsilon) \neq$$

$$P\text{-completion of } \bigcap_{\varepsilon>0} \sigma(X_s, r \leqslant s \leqslant t + \varepsilon)\},$$

$$\bar{D} = \cup D_r.$$

Then by Lemma 3, D_r and therefore $\bar{D}$ is at most countable. Fix $t \notin \bar{D}$ and let $\{r_{m(i)}\}$ be a collection of rationals increasing to t. Then for $i \geqslant 1$,

$$\bigvee_{\varepsilon>0} \sigma(X_s, r_{m(i)} \leqslant s \leqslant t - \varepsilon) = \bigcap_{\varepsilon>0} \sigma(X_s, r_{m(i)} \leqslant s \leqslant t + \varepsilon)$$

on P-completion. Thus

$$\xi(t+) \subset P\text{-completion of } \bigvee_{\varepsilon>0} \sigma(X_i, r_{m(i)} \leqslant s \leqslant t - \varepsilon), \quad i \geqslant 1,$$

and hence

$$\xi(t+) \subset P\text{-completion of } \bigcap_i \bigvee_{\varepsilon>0} \sigma(X_s, r_{m(i)} \leqslant s \leqslant t - \varepsilon) \subset \xi(t-).$$

The claim follows. □

Proof of Theorem 1. It suffices to consider $t \in [0, T]$ for some finite $T > 0$. Applying the above corollary to the process $X(T-t)$, $t \in [0, T]$, we conclude that $\xi(t-) \subset \xi(t+)$ except at most countably many t. Combine this with the corollary to conclude. □

COROLLARY 2

If $X(\cdot)$ is a Markov process, then $\xi(t+) = \xi(t-) = \xi(t)$ ($\triangleq$ the P-completion of $\sigma(X(t))$), at all but at most countably many t.

Proof. Note that $\xi(t) \subset \xi(t+) \cap \xi(t-)$ for all $t \geqslant 0$. Let $t \geqslant 0$ be such that $\xi(t+) = \xi(t-)$. Since $X(\cdot)$ is Markov, $\xi(t+)$, $\xi(t-)$ are conditionally independent given $X(t)$. Thus $\xi(t+)$ is conditionally independent of itself given $X(t)$, implying $\xi(t+) \subset \xi(t)$. Similarly $\xi(t-) \subset \xi(t)$. □

It is conjectured that the conclusions of Corollary 2 hold even in absence of the Markov property. If true, this result will have important implications in stochastic control theory [1]. We conclude with an example to show that one cannot improve on Theorem 1 in general.

Example Let $\Omega = [0, 1]^\infty$, $\mathscr{F}_0$ = the product Borel σ-field, P = the product Lebesgue measure and $\mathscr{F}$ the product σ-field completed with respect to P. Let $w = (w_1, w_2, \cdots)$ denote a typical element of Ω. Let $\{r_n\}$ be an enumeration of rationals in (0, 1). Define an R^∞-valued process $X(t) = [X_1(t), X_2(t), \cdots]$, $t \in [0, 1]$ as follows: $[X_1(t), X_2(t), \cdots]$ evaluated at the sample point $[w_1, w_2, \cdots]$ is given by

$$X_{2i}(t) = w_{2i}[(t - r_i)^+], i \geqslant 1,$$

$$X_{2i-1}(t) = w_{2i-1}[(t - r_i)^-], \quad i \geqslant 1,$$

for $t \in [0, 1]$. Then it is easy to see that $\xi(t+)$, $\xi(t-)$, $\xi(t)$ are the P-completions of $\Pi_n G_n^+(t)$, $\Pi_n G_n^-(t)$, $\Pi_n G_n^0(t)$ respectively, where $G_n^+(t)$, $G_n^-(t)$, $G_n^0(t)$ are as described below: Let $\mathfrak{B}$ = the Borel σ-field of [0, 1], β = the trivial σ-field $\{\phi, [0, 1]\}$ on [0, 1]. Then for $n \geqslant 1$,

$$G_{2n}^+(t) = \mathfrak{B} \text{ if } t \geqslant r_n, = \beta \text{ otherwise},$$

$$G_{2n-1}^+(t) = \mathfrak{B} \text{ if } t < r_n, = \beta \text{ otherwise},$$

$$G_{2n}^-(t) = \mathfrak{B} \text{ if } t > r_n, = \beta \text{ otherwise},$$

$$G_{2n-1}^-(t) = \mathfrak{B} \text{ if } t \leqslant r_n, = \beta \text{ otherwise},$$

$$G_{2n}^0(t) = \mathfrak{B} \text{ if } t > r_n, = \beta \text{ otherwise},$$

$$G_{2n-1}^0(t) = \mathfrak{B} \text{ if } t < r_n, = \beta \text{ otherwise}.$$

It follows that $\xi(t+) = \xi(t-) = \xi(t)$ for t irrational in [0, 1] whereas $\xi(t+) \neq \xi(t-) \neq \xi(t)$ for $t \in \{r_n, n \geqslant 1\}$.

Remark: One may replace $X(\cdot)$ above by a real valued process without altering the conclusions by virtue of the isomorphism theorem for Polish spaces (Theorem 2.12, p. 14, of [4]).

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Absolutely continuous spectrum for sparse potentials

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Abstract. In this paper we show that bounded perturbation of the discrete laplacian with a potential that is sparsely supported (a notion made precise in the paper) produces absolutely continuous spectrum in the interval $[-2\nu, 2\nu]$ for large dimension ν. We note that the potential need not give rise to a compact operator, let alone have decay at infinity.

Keywords. Absolutely continuous spectrum; sparse potentials.

1. Introduction

The study of the absolutely continuous spectrum is relevant in several models associated with scattering on the one hand and the extended states, as is expected in the Anderson model, etc. In our attempt to understand the mechanism of occurrence of the absolutely continuous spectrum in the Anderson model, we come across a situation where a certain non-compact perturbation of the discrete laplacian in 4 or more dimensions exhibits absolutely continuous spectrum if the potential is sparsely supported in the lattice. We may consider this as a hierarchial model also for the purposes of understanding the mechanism responsible for producing absolutely continuous spectrum in the higher dimensional Anderson model. We stress here that we only consider a deterministic operator and show the presence of absolutely continuous spectrum.

We also would like to add here that in the Anderson model the diffusive behaviour is supposed to be responsible for the occurrence of the absolutely continuous spectrum, while in this work it is scattering that produces the absolutely continuous spectrum. We would like to still present this result with the hope that it might be useful for further work in the area. We recall that in [1] it was shown that if the potential is i.i.d. random variable with distribution having finite variance times a function with a overall decay at infinity as a power $\alpha < -1$ of the distance from the origin, then the perturbation of the laplacian with such potentials has absolutely continuous spectrum in $[-2\nu, 2\nu]$ almost everywhere. In the proof of that theorem the behaviour of the kernel of the free evolution in time is used together with the conditions on the potentials to show the existence of the wave operators and thus the presence of the absolutely continuous spectrum. It was sufficient to make crude estimates to obtain the result there. In this paper we show that it is actually not necessary for the potential to have decay at infinity provided it has some sparse support, in such a way that we can still show the existence of the wave operators. But for technical reasons we stick to the deterministic case and consider only single potentials. It is clear that some random potentials of small disorder would have sparse support but not those assumed here. Therefore we cannot extend our results yet to cover them.

2. The model

We consider the space $l^2(\mathbb{Z}^\nu)$ and the laplacian Δ defined by

$$(\Delta u)(n) = \sum_{|n-i|=1} u(\mathrm{i}) \tag{2.1}$$

and a potential V coming from a real valued bounded function $V:\mathbb{Z}^\nu \to \mathbb{R}$. We then consider the operator $\mathrm{H} = \Delta + \mathrm{V}$ and note that it is self-adjoint on $l^2(\mathbb{Z}^\nu)$. We also note that the operator Δ has spectrum $[-2\nu, 2\nu]$ which is purely absolutely continuous and H has spectrum in $[-2\nu - \|V\|_\infty, 2\nu + \|V\|_\infty]$. We make the following assumption on the potential V.

Assumption 2.1 *We assume that if* Λ *is a* ν*-dimensional cubic region in* $\mathbb{Z}^\nu$ *centered at any point in* $\mathbb{Z}^\nu$, *then the volume* $|S|$ *of the set* $S = \{n \in \Lambda\colon V(n) \neq 0\}$ *satisfies*

$$|S| \leqslant |\Lambda|^{1-\eta}$$

for $|\Lambda|$ *sufficiently large. We assume that* η *satisfies* $4/\nu > \eta > 3/\nu$.

Our main theorem of the paper is the following.

Theorem 2.2 *Consider the operator H defined above on* $l^2(\mathbb{Z}^\nu)$ *and suppose the potential* V *satisfies the assumption* 2.1, *then for* $\nu \geqslant 4$ *the absolutely continuous spectrum of H contains the interval* $[-2\nu, 2\nu]$.

At this stage we would like to mention the strategy of proof. It turns out that the time evolution generated by the operator Δ has decay like $t^{-(\nu/2)}$ in ν dimensions in most of the regions of the lattice. However there is a thin region in the lattice where the decay is slightly lower. Therefore if the potential is nonvanishing in regions which are in small volume compared to the volume of the region under consideration, then the overall decay of the sum of matrix elements of the time evolution would have integrable decay, which would then given using standard techniques of scattering theory the necessary result from the existence of the wave operators.

Proof. We notice that to prove the theorem we need to show the existence of the limits $\lim_{t\to\infty} \exp(-\mathrm{i}t\Delta)f$ for each $f \in l^2(\mathbb{Z}^\nu)$. This is reduced to showing by density arguments that the sequence

$$F(\phi, f, t) = \langle \phi, \exp(itH)\exp(-it\Delta)f\rangle \tag{2.2}$$

is Cauchy in t, uniformly in ϕ and for all elements f of $l^2(\mathbb{Z}^\nu)$ of finite support.

Thus we have to show that

$$\lim_{t,w\to\infty} \sup_{\|\phi\|=1} \{|\langle \phi, \exp(itH)\exp(-it\Delta)f\rangle - \langle \phi, \exp(iwH)\exp(-iw\Delta)f\rangle|\} = 0 \tag{2.3}$$

By writing the difference on the left hand side as an integral of the differential one has to show that

$$\lim_{t\to\infty} \sup_{\|\phi\|=1} \left\{ \int_w^t \frac{d}{ds} \langle \phi, \exp(isH)\exp(-is\Delta)f\rangle \right\} = 0 \tag{2.4}$$

Computing the integrand in the above expression and simplifying we get that it is sufficient to show the integrability in s of

$$\sup_{\|\phi\|=1} \{ |\langle \phi, \exp(isH)\, V\exp(-is\Delta)f\rangle| \} \tag{2.5}$$

Since by the argument above f is a vector of finite support in the above expression, it is sufficient to consider the case when f is a vector with support at a single point in $\mathbb{Z}^\nu$ in particular the element δ_0. We shall not make use of any special properties of the point 0 of $\mathbb{Z}^\nu$, in the following. Therefore we consider,

$$\sup_{\|\phi\|=1} \{ |\langle \phi, \exp(isH)\, V\exp(-is\Delta)\delta_0\rangle| \} \tag{2.6}$$

By introducing a partition of unity in the above expression we find that it is equal to

$$\sup_{\|\phi\|=1} \left\{ \left| \left\langle \phi, \exp(isH) \sum_{n\in\mathbb{R}^\nu} \middle| \delta_n \right\rangle \langle \delta_n | V\exp(-is\Delta)\delta_0\rangle \right| \right\} \tag{2.7}$$

Since the operator $\exp(isH)$ is unitary for each fixed s, the above quantity can be estimated, using the Cauchy-Schwarz inequality, by the following quantity

$$\sup_{\|\phi\|=1} \left\{ \|\phi\|^2 \sum_{n\in\mathbb{Z}^\nu} |V(n)|^2 |\langle \delta_n, \exp(-isH)\delta_0\rangle|^2 \right\}^{1/2} \tag{2.8}$$

Therefore it is enough to show that the following estimate is valid as $s \to \infty$,

$$\sum_{n\in\mathbb{Z}^\nu} |V(n)|^2 |\langle \delta_n, \exp(-is\Delta)\delta_0 >|^2 \approx O(|s|^{-2-}) \tag{2.9}$$

To show this we need to consider the sum split into several regions in $\mathbb{Z}^\nu$ depending upon the parameter s and estimate the summand in each of these regions. We shall do this in the following lemmas and complete the proof of the theorem.

Lemma 2.3 *Consider the number β as in Proposition A.* 3 *and consider the region $S_s = \{n\in\mathbb{Z}^\nu : |n_i| > \beta_s$, for some $i = 1, \dots \nu\}$. Then the following estimate is valid.*

$$\sum_{n\in S_s} |V(n)|^2 |\langle \delta_n, \exp(-is\Delta)\delta_0\rangle|^2 \approx O(|s|^{-2-}) \tag{2.10}$$

as $s \to \infty$.

Proof. To evaluate the sum in (2.10), we note that $S_s = \cup_i \{n\in\mathbb{Z}^\nu : |n_i| > \beta|s|\} = \cup_i S_s(i)$. Therefore on each of these regions we have the estimate, coming from Propositions A.2

and A.3,

$$|\langle\delta_n, \exp(-is\Delta)\delta_0\rangle|^2 \leqslant \exp(-\alpha(\beta)|n_i|) \prod_{j\neq i} |J_{n_j}(2s)|^2 \tag{2.11}$$

Therefore we have the estimate

$$\sum_{n\in S_s} |V(n)|^2 |\langle\delta_n, \exp(-is\Delta)\delta_0\rangle|^2$$

$$\leqslant \sum_{i=1}^{\nu} \sum_{S_s(i)} |V(n)|^2 \exp(-\alpha(\beta)|n_i|) \prod_{j\neq i} |J_{n_j}(2s)|^2 \tag{2.12}$$

Now $V(n)$ is a bounded function, $|n_i| > \beta|s|$ and $n_j, j\neq i$ is in $\mathbb{Z}$. Therefore the exponential term in $|n_i|$ gives a decay in s of the required order for the sum since $J_{n_j}(2s)$ is in $l^2(\mathbb{Z})$ with $\Sigma_{n_j}|J_{n_j}(2s)|^2$ uniformly bounded in s.

Now it remains to show that in the region $\mathbb{Z}^\nu - S_s$ the decay at infinity in s required by equation (2.9) is valid. Infact in the above Lemma we did not make use of the Assumption 2.1 at all. It is in the other region that we need the assumption as will be seen in the proof of the following lemma.

Lemma 2.4 *Consider the region* $\mathbb{Z}^\nu - S_s, S_s$ *as in Lemma* 2.3. *Then in this region the following estimate is valid.*

$$\sum_{n\in\mathbb{Z}^\nu - S_s} |V(n)|^2 |\langle\delta_n, \exp(-is\Delta)\delta_0\rangle|^2 \approx O(|s|^{-2-}) \tag{2.13}$$

as $|s| \to \infty$.

Proof. The region $\mathbb{Z}^\nu - S_s$ is nothing but the region $\{n\in\mathbb{Z}^\nu : |n_i| \leqslant \beta|s|, \forall i = 1,\ldots,\nu\}$, let us call it S_s^c. We split the region S_s^c further into the following regions.

$$S_s^c = \cup_{k=0}^{\nu} S_s^c(k)$$

and

$$S_s^c(k) = S_s^c \cap \{|n_i - 2s| < \sqrt{s}, i = 1,\ldots,k\} \cap \{|n_i - 2s| \geqslant \sqrt{s}, i = k+1,\ldots \nu\} \tag{2.14}$$

where $k = 0$ is to be understood as the absence of the second factor. We shall show that in each of the above regions we have the estimate claimed in the lemma for the left hand side of equation (2.13). All the constants that occur in the following estimates would be independent of s, but may depend on the dimension ν, the point 0 of $\mathbb{Z}^\nu$, on $\|V\|$ and the exponent η of assumption 2.1. Now let us consider first the region $S_s^c(0)$. Then we have, as $s \to \infty$, by Propositions A.2 and A.4 (1),

$$\sum_{n\in S_s^c(0)} |V(n)|^2 |\langle\delta_n, \exp(-is\Delta)\delta_0\rangle|^2 \leqslant C\ Vol\{n\in S_s^c(0)\colon V(n) \neq 0\} |s|^{-\nu} \tag{2.15}$$

where Vol (A) denotes the cardinality of the set A. Now by assumption 2.1, the Vol in the above inequality is estimated as

$$Vol(n\in S_s^c(0)\colon V(n) \neq \{Vol(S_s^c(0))\}^{1-\eta} \leqslant \text{Const.}|s|^{\nu(1-\eta)} \tag{2.16}$$

where the Const. is independent of s. Combining equations (2.15) and (2.16) we see that

$$\sum_{n\in S_s^c(0)} |V(n)|^2 |\langle \delta_n, \exp(-is\Delta)\delta_0\rangle|^2 \leqslant C|s|^{-\nu\eta} \tag{2.17}$$

Now η being bigger than $3/\nu$, by assumption 2.1, the required estimate follows for the sum in this region. As for the other region we shall estimate for a general k. We have the Propositions A.2 and A.4 (1), (2) that

$$\begin{aligned}&\sum_{n\in S_s^c(k)} |V(n)|^2 |\langle \delta_n, \exp(-is\Delta)\delta_0\rangle|^2 \\ &\leqslant C\ Vol\{n\in S_s^c(k)\colon V(n)\neq 0\}|s|^{-(\nu-k)}|s|^{-(2k/3)}\end{aligned} \tag{2.18}$$

and the estimate for the Vol in the above inequality for each k is done as follows. We consider a region $S_s^c(k)$ and cover it by disjoint cubic regions of sides $\sqrt{s}$. Then, since in each of such cubic regions the set of points where V does not vanish scales as $(1-\eta)$ of the volume, and since there are about $|s|^{(1/2)(\nu-k)}$ such cubes covering the region $S_s^c(k)$ we have the estimate that for each $k=1,\dots,\nu$,

$$Vol(\{n\in S_s^c(k)\colon V(n)\neq 0\}) \leqslant \text{Const.}\ |s|^{(\nu/2)(1-\eta)}|s|^{(\nu-k)/2} \tag{2.19}$$

On the other hand for some values of k the above estimate could turnout to be a gross over estimate of the actual volume of the points where V is non zero. Therefore we also consider the estimate that

$$Vol(\{n\in S_s^c(k)\colon V(n)\neq 0\}) \leqslant Vol(\{n\in S_s^c\colon V(n)\neq 0\}) \leqslant \text{Const.}\ |s|^{\nu(1-\eta)} \tag{2.20}$$

From equations (2.18)–(2.20), it is clear that the decay as $s\to\infty$ claimed in lemma is valid if, for each $k=1,\dots\nu$ we have that

$$\min\left\{\frac{(\nu-k)}{2}+\nu\frac{(1-\eta)}{2}, \nu(1-\eta)\right\}+\left(-\nu+k-\frac{2k}{3}\right)<-2 \tag{2.21}$$

This inequality simplifies to the following

$$-\eta-\max\left\{\eta,\frac{k}{\nu}\right\}+\frac{2}{3}\frac{k}{\nu}<-\frac{4}{\nu} \tag{2.22}$$

This equation is satisfied for each $k=1,\dots,\nu$ if η is chosen so that $4/\nu>\eta>3/\nu$, when $\nu\geqslant 4$. Thus under the assumption 2.1, the lemma is proved.

We shall give an example of a potential V satisfying assumption 2.1, below.

Example. Let us consider $\mathbb{Z}^4$ and the S defined by

$$S=\{n\in\mathbb{Z}^4\colon \forall i=1,\dots,4, n_i=k_i^5 \text{ for some } k_i\in\mathbb{Z}\}$$

Now consider any real valued function V which takes non-zero values on S and is zero in $\mathbb{Z}^4-S$. It is not difficult to check that such a V satisfies the assumption 2.1, with $\eta=4/5$.

One can construct in a similar fashion a host of example of functions in any dimension bigger than 3. It is also clear that if V is bounded but does not vanish one any point of S, it cannot give rise to a compact operator in $l^2(\mathbb{Z}^\nu)$. Of course it goes without saying that if the potential consists of two parts one satisfying the assumptions given in assumption 2.1 and the other having a decay at infinity as in [1], the Theorem 2.2 will be valid for the sum.

Appendix

In the appendix we collect a few propositions that are standard and which are used in the paper.

PROPOSITION A.1

If A and B are two bounded self-adjoint operators on a Hilbert space and suppose the limits $\mathrm{slim}_{t\to\infty} \exp(iAs)\exp(-iBs)E_{ac}(B)$ *exists. Then* $\sigma_{ac}(B) \subset \sigma_{ac}(A)$.

The above theorem is standard and the proof can be found in any standard work on scattering theory, for example [2] [3].

PROPOSITION A.2

Let $m, n \in \mathbb{Z}^\nu$ and Δ be as in (2.1). *Then we have*

$$\langle \delta_n, exp(-i\Delta s)\delta_m \rangle = \prod_{i-1}^{\nu} (-i)^{(n-m)_i} J_{(n-m)_i}(2s) \tag{A.1}$$

where $J_k(s)$ is the Bessel function of integral order k.

The expression for the matrix elements of the free evolution in terms of the Bessel functions follows from direct verification, see for example the Appendix of [1].

PROPOSITION A.3

Consider $J_n(x), n \in \mathbb{Z}$ and $x > 0$. Then $\exists \beta > 0$ and $\alpha(\beta) > 0$ such that

$$|J_n(x)| \leqslant C\exp(-\alpha(\beta)|\eta|) \tag{A.2}$$

whenever $|\eta| > \beta|x|$.

Proof. We have the following relation for the Bessel function of integral order. $J_n(x) = (-1)^n J_{-n}(x)$ for n a positive integer. Therefore we consider only positive n in the following. Now the power series representation of $J_n(x)$, we have that

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m(\frac{1}{2}x)^{n+2m}}{m!(n+2m)!}$$

From this it follows that

$$|J_n(x)| \leqslant \frac{\frac{1}{2}|x|^n}{n!}\exp\left(\frac{1}{4}\frac{|x|^2}{(n+1)}\right)$$

Now we have the Stirling's approximation for the factorial n, when $n \to \infty$, as

$$n! \approx \frac{1}{\sqrt{2\pi n}} \exp(-n) n^n$$

Hence, for some constant C,

$$|J_n(x)| \leqslant C\sqrt{2\pi n} \exp\left\{ n\left[\log\left(\frac{|x|e}{2n}\right) + \frac{|x|^2}{4n(n+1)} \right]\right\}$$

Therefore if $|n| > e^2|x|/2$ we can easily check that

$$|J_n(x)| \leqslant C\sqrt{2\pi n} \exp\left\{ \left(-1 + \frac{1}{e^4} \right) n \right\} \leqslant C \exp(-\alpha(\beta)n)$$

for $\beta = \frac{1}{2}e^2$ and an appropriate α.

PROPOSITION A.4

Consider $J_n(x)$ for x and n both large. Then we have the following behaviour as $x \to \infty$, in the regions indicated below.

(1) *If* $|x - n| \geqslant |x|^{1/2}$, *then* $|J_n(x)| \leqslant C|x|^{-1/2}$.

(2) *If* $|x - n| \leqslant |x|^{1/2}$, *then* $|J_n(x) \approx O(|x|^{-1/3})$.

In both the cases the error involved would be $|x|^{-1}$.

The above proposition is well known classically and is proved using the stationary phase techniques and the details can be found in Watson [4].

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Oscillation of odd order delay differential equations

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Abstract. In this paper, we obtain the sufficient and necessary conditions for all solutions of the odd-order nonlinear delay differential equation.

(*) $x^{(n)} + Q(t)f(x(g(t))) = 0$

to be oscillatory. In particular, if $n = 1$, $Q(t) > 0$, $f(x) = x^{\alpha}$, where $\alpha \in (0, 1)$ and is a ratio of odd integers and $g(t) = t - \tau$ for some $\tau > 0$, then every solution of (*) oscillates if and only if $\int^{\infty} Q(s)\mathrm{d}s = \infty$.

Keywords. Odd order; retarded nonlinear differential equations; oscillation of all solutions.

1. Introduction

In this paper we consider the *odd order* delay differential equation

(E) $$x^{(n)}(t) + Q(t)f(x(g(t))) = 0,$$

with the assumptions that

(H) $$\begin{cases} Q \in C([0,\infty),[0,\infty)), f, g \in C(R,R) \text{ such that } f \text{ is increasing, } xf(x) > 0 \text{ for } x \neq 0, \\ g(t) < t,\ g(t) \to \infty \text{ as } t \to \infty \text{ and } n \text{ is an odd integer.} \end{cases}$$

Let $\phi \in C([g(T), T], R)$. By a solution of (E) in $[T, \infty)$, we mean a function $x \in C^{(n)}([T, \infty), R)$ such that $x(t) = \phi(t)$ for $g(T) \leqslant t \leqslant T$ and satisfies (E) for $t \geqslant T$.

A solution $x(t)$ of (E) is said to be oscillatory if it has zeros for arbitrarily large t. Otherwise, we call it nonoscillatory.

We call that (E) is generalized sublinear if f satisfies

(C$_1$) $$0 < \int_0^{\pm K} \frac{\mathrm{d}x}{f(x)} < \infty \text{ for every } K > 0,$$

which includes the case $f(x) = x^{\alpha}$, $0 < \alpha < 1$ and α is a ratio of odd integers. Similarly, (E) is said to be generalized superlinear, if f satisfies

$$0 < \int_{\pm K}^{\infty} \frac{\mathrm{d}x}{f(x)} < \infty \text{ for every } K > 0,$$

which includes the case $f(x) = x^{\alpha}$, $\alpha > 1$ and α is a ratio of odd integers.

In what follows, we list some important results relevant to this paper. Consider the first order delay differential equations

$$(\mathrm{E}_0) \qquad x'(t)+Q(t)x(g(t))=0$$

and

$$(\mathrm{E}_1) \qquad x'(t)+Q(t)f(x(g(t)))=0,$$

where Q, f and g are as defined above.

(R_1). Ladas [4] (see also [5]). All solutions of (E_0) are oscillatory if

$$\liminf_{t\to\infty}\int_{g(t)}^{t} Q(s)\mathrm{d}s>\frac{1}{e}.$$

(R_2). Ladde *et al* [5]. All solutions of (E_0) are oscillatory implies that

$$\limsup_{t\to\infty}\int_{g(t)}^{t} Q(s)\mathrm{d}s>\frac{1}{e}.$$

(R_3). Shreve [6]. If $\lim_{x\to 0} x/f(x)=M$, then all solutions of (E_1) are oscillatory if

$$(\mathrm{C}_2) \qquad \limsup_{t\to\infty}\int_{g(t)}^{t} Q(s)\mathrm{d}s>M.$$

(R_4). Graef *et al* [3]. If f satisfies (C_1), then all solutions of (E_1) are oscillatory if

$$(\mathrm{C}_3) \qquad \int^{\infty} Q(s)\mathrm{d}s=\infty.$$

(R_5). Das [1]. If $f(x)=x^{\alpha}, 0<\alpha<1$ and α is a ratio of odd integers, then all solutions of (E_1) are oscillatory *if and only if* (C_3) holds.

From (R_1) and (R_2), it follows that when $Q(t)=q\in(0,\infty)$ and $g(t)=t-\sigma$ for some $\sigma>0$, then all solutions of (E_0) are oscillatory if and only if $q\sigma>1/e$. Thus, the gap between (R_1) and (R_2) remains as an open question. Clearly, (R_3) and (R_1) are different. We may easily see that (C_3) is a weaker condition than that of (C_2). Again $\lim_{x\to 0} x/f(x)=M$ and f is continuous implies that $x/f(x)$ is bounded in any finite neighbourhood of zero. Hence for every $K>0$, there exists a $\mu_K>0$ such that

$$\frac{x}{f(x)}\leqslant\mu_K \text{ for } x\in[-K,K].$$

Clearly,

$$\begin{aligned} I=\int_0^K\frac{x}{f(x)}\mathrm{d}x&=K\int_0^K\frac{\mathrm{d}x}{f(x)}-\int_0^K\left(\int_0^t\frac{\mathrm{d}x}{f(x)}\right)\mathrm{d}t\\ &=K\int_0^K\frac{\mathrm{d}x}{f(x)}-\int_0^K\frac{x}{f(x)}\mathrm{d}x\\ &=K\int_0^K\frac{\mathrm{d}x}{f(x)}-I, \end{aligned}$$

that is,

$$\int_0^K\frac{\mathrm{d}x}{f(x)}=\frac{2I}{K}\leqslant 2\mu_K.$$

Hence the sublinearity condition in (R_4) is weaker than that of (R_3). Although we consider the particular case $f(x)=x^{\alpha}$ in (R_5), our result is very interesting in the sense that it is both necessary and sufficient.

In this paper we attempt to extend (R_5) for a general odd order delay differential equation of the type (E), where we consider the more general sublinear condition (C_1) in place of its particular case as given in (R_5).

2. Main results

In the beginning of this section, we prove a lemma for its use in the sequel.

Lemma 1. *Suppose that* $f\in C^{(n)}([T,\infty),(0,\infty))$ *such that* $f^{(i)}(t)$ *has no zeros in* $[T,\infty)$ $(i=1,2,\ldots(n-1))$ *and* $f^{(n)}(t)\leqslant 0$ *for* $t\geqslant T$. *If* $\sigma\in C([T,\infty),(0,\infty))$ *and bounded, then*

$$\mathrm{f}(t-\sigma(t))\geqslant\frac{(\sigma(t))^{n-1}}{(n-1)!}f^{(n-1)}(t),\quad t\geqslant T+2\sigma(t). \tag{1}$$

Proof. Since $f^{(n)}(t)\leqslant 0, t\geqslant T$, there exists an integer $k(0\leqslant k\leqslant n-1$ and k is even iff n is odd) such that $f^{(j)}(t)>0$ for $j\leqslant k$ and $f^{(j)}(t)f^{(j+1)}(t)\leqslant 0$ for $k\leqslant j\leqslant n-1$. Expanding $f(t)$ by Taylor's series, there exists $x\in(t-\sigma(t),t)$ such that

$$\begin{aligned}f(t)&=\sum_{j=0}^{k-1}\frac{\sigma^j(t)}{j!}f^{(j)}(t-\sigma(t))+\frac{\sigma^k(t)}{k!}f^{(k)}(x)\\&\geqslant\frac{\sigma^k(t)}{k!}f^{(k)}(t), t\geqslant T+\sigma(t).\end{aligned} \tag{2}$$

Expanding $f^{(k)}(t)$ by Taylor's series we get

$$f^{(k)}(t-\sigma(t))\geqslant\frac{(-\sigma(t))^{n-k-1}}{(n-k-1)!}f^{(n-1)}(t), t\geqslant T+\sigma(t). \tag{3}$$

Replacing t by $t-\sigma(t)$ in (2), then using the resulting inequality with (3) and the fact that $k!(n-k-l)!\leqslant(n-1)!$ for every integer $k(0\leqslant k\leqslant n-1)$ we get our proposed inequality.

Theorem 1. *Suppose that* (H) *holds and* f *has the property that* $f(xy)\geqslant f(x)f(y)$ *for any two continuous functions of* x *and* y. *Further, if* (C_1) *holds*, $|t-g(t)|<M$ *for some* $M>0$ *and*

$$\int^{\infty}Q(s)f((s-g(s))^{n-1})ds=\infty, \tag{4}$$

then every solution of (E_1) *oscillates.*

Proof. Contrary to the theorem, suppose that $x(t)$ is a nonoscillatory solution of (E_1). Without any loss of generality, assume that $x(t)>0, t\geqslant t_0$. Let $t_1\geqslant t_0$ such that $g(t)>t_0$ for $t\geqslant t_1$. From (E_1) it follows that $x^{(n)}(t)\leqslant 0$ for $t\geqslant t_1$. Consequently, there exists a $T \geqslant t_1$ such that $x(t)$ satisfies the hypotheses of Lemma 1. Clearly, $x^{(n-1)}(t) > 0, t \geqslant T$. By Lemma 1 we get

$$x(g(t)) \geqslant \frac{(t-g(t))^{n-1}}{(n-1)!} x^{(n-1)}(t), t \geqslant T+2M. \tag{5}$$

Hence from (5) we obtain

$$f(x(g(t))) \geqslant \lambda f((t-g(t))^{n-1}) f(x^{(n-1)}(t)), t \geqslant T+2M, \tag{6}$$

where

$$\lambda = f\left(\frac{1}{(n-1)!}\right).$$

Using (6) in (E) we get

$$x^{(n)} + \lambda Q(t) f((t-g(t))^{n-1}) f(x^{(n-1)}(t)) \leqslant 0, t \geqslant T+2M. \tag{7}$$

Dividing (7) throughout by $f(x^{(n-1)}(t))$ and integrating the resulting inequality from $T_0 = T+2M$ to t we get

$$-\int_{x^{(n-1)}(t)}^{x^{(n-1)}(T_0)} \frac{\mathrm{d}\theta}{f((\theta)} + \lambda \int_{T_0}^{t} Q(\theta) f((\theta - g(\theta))^{n-1}) \mathrm{d}\theta \leqslant 0, t \geqslant T_0. \tag{8}$$

Letting $n \to \infty$ in (8) and using (C_1) and (4) we reach to a contradiction. This proves our theorem.

COROLLARY 1

If $Q \in C([0,\infty), [0,\infty))$, $\tau > 0$, $\alpha \in (0,1)$ *is a ratio of odd integers, n is odd and*

$$\int^{\infty} Q(s)\mathrm{d}s = \infty,$$

then every solution of

$$x^{(n)} + Q(t) x^{\alpha}(t-\tau) = 0,$$

oscillates.

Its proof follows from Theorem 1.

Remark 1. It is impossible to extend Theorem 1 for linear ($f(x) = x$) and superlinear ($f(x) = x^{\alpha}, \alpha > 1$) equations. The following example strengthens our remark.

Example 1. Consider the equation

$$x^{(n)}(t) + \exp((\alpha-1)t - \alpha\sigma) x^{\alpha}(t-\sigma) = 0, \alpha \geqslant 1, \sigma > 0.$$

This equation satisfies the hypotheses of Theorem 1 (except $\alpha < 1$) and admits a nonoscillatory solution $x(t) = e^{-t}$.

Theorem 2. *Suppose that* (H) *holds and f satisfies the Lipschitz condition on R. Then* *all solutions of* (E) *are oscillatory implies that*

$$\int^{\infty} s^{n-1} Q(s)\mathrm{d}s = \infty.$$

Proof. If possible, assume that

$$\int^{\infty} s^{n-1} Q(s)\mathrm{d}s < \infty.$$

Let T is large enough such that

$$K \int_{g(T)}^{\infty} s^{n-1} Q(s)\mathrm{d}s < p < 1, \tag{9}$$

where K is the Lipschitz constant. Consider the sequence of functions $\langle x_n(t)\rangle_{n=0}^{\infty}$ in $[g(T), \infty)$ defined by

$$x_0(t) = 0, t \geqslant g(T)$$

$$x_{m+1}(t) = \begin{cases} 1 + \int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} Q(s) f(x_m(g(s)))\mathrm{d}s, t \geqslant T. \\ x_{m+1}(T), t \in [g(T), T]. \end{cases} \tag{10}$$

Clearly, $x_m(t) > 1 (m = 1, 2, \ldots), t \geqslant g(T)$. Again for $t \geqslant g(T)$.

$$|x_2(t) - x_1(t)| \leqslant \int_T^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} Q(s) |f(x_1(g(s))) - f(x_0(g(s)))| \mathrm{d}s. \tag{11}$$

Clearly, $f(x_0(s)) = f(0) = 0$ and $x_1(s) = 1$. Since f is Lipschitzian,

$$|f(u) - f(v)| < K|u - v|, \forall u, v \in R.$$

Hence

$$|f(x_1(g(s))) - f(x_0(g(s)))| \leqslant K. \tag{12}$$

Consequently, from (9), (11) and (12) we get

$$|x_2(t) - x_1(t)| < p.$$

Suppose that for some index $m \geqslant 1$,

$$|x_{m+1}(t) - x_m(t)| < (p)^m, t \geqslant g(T). \tag{13}$$

Now

$$\begin{aligned} |x_{m+2}(t) - x_{m+1}(t)| &\leqslant \int_T^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} Q(s) |f(x_{m+1}(g(s))) - f(x_m(g(s)))| \mathrm{d}s \\ &\leqslant K \int_T^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} Q(s) |x_{m+1}(g(s)) - x_m(g(s))| \mathrm{d}s \\ &\leqslant p^{m+1}, t \geqslant g(T). \end{aligned}$$

Hence by induction, (13) holds for every positive integer m. Clearly, the uniform convergence of the sequence of continuous functions $\langle x_m(t)\rangle_{m=0}^{\infty}$ follows from the uniform convergence of the infinite series

$$x_0(t)+[x_1(t)-x_0(t)]+[x_2(t)-x_1(t)]+\dots$$
$$+[x_m(t)-x_{m-1}(t)]+\dots,$$

which is dominated by the convergent geometric series $\Sigma_{n=0}^{\infty}p^n$. Let $x(t)=\lim_{m\to\infty}x_m(t)$, $t\geqslant T$. Clearly, $x(t)\geqslant 1$ for $t\geqslant T$. Further

$$|f(x_m(t))-f(x_{m-1}(t))|\leqslant K|x_m(t)-x_{m-1}(t)|$$

implies that $f(x_m(t))$ converges to $f(x(t))$ uniformly in $[T,\infty)$. Taking limit as $m\to\infty$ in (10) and using the uniform convergent theorem we get

$$x(t)=1+\int_t^{\infty}\frac{(s-t)^{n-1}}{(n-1)!}Q(s)f(x(g(s)))\mathrm{d}s. \tag{14}$$

But (14) shows that $x(t)$ is a nonoscillatory solution of (E), a contradiction to our assumption.

This completes the proof of our theorem.

COROLLARY 2

If $Q\in C([0,\infty),[0,\infty))$, $\tau>0$, $\alpha\in(0,1)$ *is a ratio of odd integers and n is odd, then all solutions of*

$$x^{(n)}+Q(t)x^{\alpha}(t-\tau)=0$$

are oscillatory implies that

$$\int^{\infty}s^{n-1}Q(s)\mathrm{d}s=\infty.$$

The proof of this corollary follows from Theorem 2.

The following corollary follows from Corollary 1 and Corollary 2.

COROLLARY 3

If Q, τ *and* α *satisfy the conditions of Corollary* 2, *then all solutions of the first order delay differential equation*

$$x'(t)+Q(t)x^{\alpha}(t-\tau)=0$$

are oscillatory if and only if

$$\int^{\infty}Q(s)\mathrm{d}s=\infty.$$

Remark 2. Corollary 3 is some as (R_5). Hence Theorems 1 and 2 generalize (R_5) mentioned here.

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